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Third Edition.

A TREATISE ON ALGEBRA.

By B. SESTINI, S. J.

Author of Analytical Geometry, and Elementary Algebra.

PROFESSOR IN GEORGETOWN COLLEGE.

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PREFACE.

This treatise may be considered as a sequel to the small Elementary Algebra, whose second edition, revised and enlarged, has just preceded the present publication. However, it is not so connected with the Elementary Algebra that it might not be taken alone, for it does not depend on the former in any of its parts, and is complete, as far as is allowed by the nature of a book destined for the use of those who desire to be initiated in the study of Algebra.

The reader, even before perusing the present introduction, has probably noticed the difference of type, intended to separate those subjects which are more accessible to pupils at large from those which suppose in the student either quicker parts or already some advancement in the study of Algebra. That is to say, the most elementary principles adapted even for those who, for the first time, open a book of Algebra, are printed in larger type: the other parts, which enter a little more into the secrets of the science, are printed in smaller characters.

We beg the reader, however, to observe, first, that the understanding even of the most elementary principles of Algebra and Geometry supposes always a certain degree of aptitude. Of this, one who for any time has had experience of the tedious labour of teaching, will render, without hesitation, abundant testimony. Another observation to be made is, that the separation adopted in the present treatise, with distinction of type, does not trace a limit to be scrupulously followed, so that the teacher or the student be compelled to go over all that is printed in large characters before commencing the rest. But it is left to the discretion of the teacher to enter, more or less, into the subject where and when he will judge fit to do so. The teacher is fully aware that he must unquestionably labour, and must not be satisfied merely with what he is to teach, but he should know much more. He should be master of the subject, and be competent to adapt it to the capacity of his pupils.

The Treatise is divided into two parts, the first of which contains algebraical operations, with several questions and doctrines connected with them, so that each section may prove complete in its own subject, and the inconvenience of returning elsewhere to speak of matter left unfinished before, may be avoided. The same method is followed in the second part, of which we will immediately say a few words. With this method, every thing is put in its own place, so that any one who would go over the whole uninterruptedly might have the advantage of order, and of seeing, at a single glance, all that each subject embraces. Nay, the same advantages may be enjoyed by those also who will be able to overcome the first difficulties at a second or third reading. This method, we believe, has also the advantage of contracting the bulk of the volume, which the same subjects, disorderly scattered, would render much larger.

The second part contains the most indispensable theories of equations, proportions, and progressions, logarithms, and some few principles on the series. The doctrine of equations has been treated more copiously than the others, not so much on account of its importance, as because it is well adapted to give an idea of algebraic analysis, and thus prepare the mind of the student who would afterwards apply himself to higher studies.

GEORGETOWN COLLEGE, *July*, 1855.

CONTENTS.

PAGE
INTRODUCTORY ARTICLE 13
Mathematics: its object, and various branches 13
Algebra: its object and generality—its symbols 13
Numerical relations of quantities 13
Relation of oppositions and signs 14
Opposite quantities—how mutually influenced 14
Coefficient and similar terms 15
Signs of addition and subtraction 15
Equations and inequalities 15
Monomials and polynomials 16
Members of equations 16
Constant and variable quantities.—Functions 16
Modification of quantities—their mutual relations 17
Division of the treatise 18

FIRST PART.

ALGEBRAIC OPERATIONS.

CHAPTER I.—DEFINITIONS AND OPERATIONS ON MONOMIALS.

ART. 1. *Addition and Subtraction.—Addition* 19
Algebraic and arithmetical addition compared 19
Definition of algebraic addition 20
Different cases 20
Rules 1 and 2 20
Doctrine of signs—product of signs 21
Examples and problems 22
Letters used for unknown quantities 24
Numerical values and general character of algebraical questions 24
Subtraction.—Difference—definition—result of signs 25
Rule and general examples 26

PAGE
Criterion of magnitude........ 26
Twofold acceptation of numbers........ 27
The term zero taken as a mere term of comparison........ 27
Examples and problems........ 27
ART. 2. *Multiplication and Division.—Multiplication*........ 28
In what multiplication consists—nomenclature........ 28
Various manners of representing the product........ 29
Definition of numerical multiplication........ 29
Case of the multiplier, whole number, and positive........ 29
Consequence concerning the sign of product........ 29
Case of the multiplier negative........ 30
Case of the multiplier fractional........ 30
Case of the irrational factors........ 30
Arithmetical rule applicable to quantities........ 30
Mutual influence of factors........ 31
Rule of signs........ 31
Various forms of the numerical value of quantities........ 31
General formulas of numerical multiplication........ 32
How the arithmetical rules of multiplication follow from the definition 32
Mechanical artifice, showing the aliquot part of a fraction........ 33
The multiplicand may become the multiplier, and *vice versa*, without affecting the product........ 33
Factors, whole numbers, and other cases........ 34
General inference........ 34
The product of any number of terms is the same, whatever be the order of the factors........ 34
Sign to be given to the product of several factors—three cases........ 36
Rule........ 37
Product of repeated factors—exponent........ 37
Examples and problems........ 38
Division.—Its definition........ 40
Algebraical expression of division........ 40
Rule—numerical values........ 41
Compound monomials........ 42
Their product and quotient........ 43
Reduction of results to a simpler form........ 44
Reduction to the same denominator........ 44
Examples and problems........ 45
ART. 3. *Formation of Powers and Extraction of Roots*........ 47
Powers.—What a power is—root, exponent, degree, power........ 47
Formation of powers, embracing all cases of numerical values, its description and definition........ 47

PAGE
Apparent powers.. 48
Powers of unity.. 48
Product of powers—case of the exponent's whole numbers.............. 49
" " case of the exponent's fractional numbers.......... 49
General inference and rule.. 50
Product of powers when the same exponent is applied to different roots 50
Inference and rule.. 50
Powers of powers.. 50
Inference and rule.. 52
Simplification of exponential fractions—different cases—examples.... 52
Extraction of roots is the inverse operation of raising to a power..... 55
Signs and nomenclature.. 56
The root of a quantity may be represented by a fractional exponent given to the same quantity.. 56
Fractional index of the root.. 56
A radical having a fractional index can be transformed into one having a whole number for its index.. 57
Product of roots—different cases.. 57
The index of the root and that of the quantity under the radical sign may be multiplied or divided by the same quantity or number.... 58
Different cases embraced by a radical expression.. 58
Imaginary radicals or roots.. 59
Operations on imaginary quantities.. 59
Irrational radicals and quantities.. 61
Greatest common measure of numbers.. 61
Prime numbers.. 63
How a prime number can accurately divide the product of two other numbers.. 64
A prime number dividing a product, divides at least one of the factors 65
Powers of fractional expressions.. 66
Irrational roots.. 67
A series of rational or assignable numbers may be conceived approaching constantly to any irrational root.. 68
Miscellaneous examples.. 69

CHAPTER II.—OPERATIONS ON POLYNOMIALS.

Art. 1. *Addition and Subtraction*.. 73
Reduction of similar terms—rule and examples.. 73
Subtraction.—Rule—examples.. 75
Art. 2. *Multiplication and Division*.. 76
Rule and examples.. 76
Rule of signs demonstrated.. 77

PAGE
Remark on common factors........ 78
Examples of multiplication........ 78
Singular property of numbers........ 80
Division.—Polynomials arranged........ 81
Operation explained........ 81
Rule and examples........ 81
Remark concerning the arrangement of polynomials........ 84
The dividend not exactly divisible by its divisor........ 85
Number of the terms of the indefinite quotient........ 87
The quotient of the same form as the imaginary expressions divided by one another........ 89
ART. 3. *Formation of Powers and Extraction of Roots*........ 90
Newtonian formula........ 91
General term........ 92
Square and cube of any binomial........ 92
Binomial theorem demonstrated........ 93
Singular property of the coefficients of the binomial........ 94
Permutations........ 94
Corollary........ 97
Combinations........ 98
On some property of numbers—theorem........ 99
Evolution of the binomial when the exponent is negative........ 100
Extraction of Roots.—The root of a polynomial another polynomial. —Example........ 102
The process to find the square root of any polynomial is always the same........ 103
General rule for the extraction of square roots of polynomials.—Examples........ 105
Extraction of square roots of numbers........ 106
Practical rules........ 107
Polynomial expression of the square of a number........ 110
Extraction of cubical roots of polynomials........ 112
Extraction of cubical roots of numbers........ 114

SECOND PART.

ALGEBRAIC THEORIES.

CHAPTER I.—EQUATIONS.

Division of the chapter........ 116
Practical rules........ 117
Known and unknown quantities........ 118
Determinate and indeterminate equations........ 119

PAGE
ART. 1. *Equations of the first degree* 119
General formula of determinate equations of the first degree 120
Resolution of any determinate equation of the first degree.—Rule.. 120
Examples and problems 121
Indeterminate equations of the first degree 122
Equations containing several unknown quantities 123
Different methods of resolution.—First, by comparison 124
Elimination by substitution 125
Elimination by addition and subtraction 127
The same methods applicable to all cases 128
Incompatible equations 128
Rules and examples 129
Problems—general rule 131
ART. 2. *Equations of the second degree* 135
General formula of the determinate equations of the second degree.. 135
General rule—remark 136
Resolution of the general equation 137
First case.—Rule and examples 137
Second case 138
Real and imaginary roots 139
Examples and problems 141
Rule 142
ART. 3. *On some properties of determined equations of any degree* 144
Preliminary theorems 144
General formula of a determined equation of any degree 147
Equations resolvable with at least one real root 147
Equations resolvable with at least two real roots 147
When equations cannot be resolved with real roots, they are resolvable with one or more imaginary roots 148
Any determined equation of any degree admits of as many roots as there are units in the degree of the equation 149
Connection between the roots and the coefficients of any determined equation 151
Corollaries.—Case of one or more roots equal to zero 153
Signs of the roots changed 153
Roots multiplied and denominators eliminated 153
The sums of various powers of the roots can be known, although the roots themselves are unknown 155
Corollary and criterion 158
ART. 4. *Resolution of determinate equations of the third and fourth degree, having real coefficients* 161
General formulas of the equations of the third degree 161

PAGE
Roots of the general equation of the third degree........................ 161
Resolution of the same general equation.. 163
The first condition always fulfilled, but not the second.................... 164
Examples.. 167
General remark.. 168
Equations of the fourth degree.. 169
Quality of the roots: how found out.. 171
Example... 172

CHAPTER II.—RATIOS, PROPORTIONS, AND PROGRESSIONS.

Division of the chapter, and definitions.. 173
ART. 1. *Arithmetical Ratios, Proportions, and Progressions*............ 173
Ratios.—Definitions and property... 173
Proportions.—Simple arithmetical proportions.............................. 174
Continual and compound proportions.. 175
Progressions.—Terms of an unlimited progression.......................... 175
General formula... 176
Sum—Examples 177
How and when the terms of a progression may be found—Example 178
ART. 2. *Geometrical Ratios, Proportions, and Progressions*............. 178
Ratios.—Definitions and property.. 178
Variable ratios—direct and reciprocal terms.................................... 179
Continual geometrical ratios—Theorem... 180
Direct and reciprocal compound ratios—Theorem............................ 181
Proportions.—Simple geometrical proportions................................ 182
General properties... 183
Continual and compound proportions.. 183
Other properties of geometrical proportions..................................... 184
Numerical proportions, the first ratio of which is irreducible............ 186
Progressions.—Terms of any geometrical progression.................... 186
General formula... 187
Sum of n terms and sum of an indefinite number of terms............. 187
Examples... 188
How and when the terms of a certain progression may be found...... 189
Example... 189

CHAPTER III.—LOGARITHMS.

Exponential quantities.. 190
Logarithms.. 190
Base and sign 191
The logarithm of unity and the logarithm of the base are the same in all systems.. 191
Positive and negative logarithms.. 192

PAGE
Theorems.. 192
In any system when the logarithms form an arithmetical progression, the corresponding numbers form a geometrical one.......... 193
Useful theorems.. 193
Common or ordinary tables of logarithms.................................. 195
Constant ratio of logarithms in every system............................. 197
How the logarithms of one system may be inferred from the logarithms of other systems.. 197
Explanatory remarks—Rules.. 198
Characteristic.. 201
Application of logarithms.. 203

CHAPTER IV.—SERIES.

What algebraic series are, and their various orders..................... 206
Various questions concerning the series.................................... 207
General term of any series.. 208
Sum of any number of terms.. 209
Examples.. 210
Problems.. 213

TREATISE ON ALGEBRA.

INTRODUCTORY ARTICLE.

Mathematics: their object.

§ 1. MATHEMATICS treat of quantities, namely, of all that which can be numerically estimated or measured.

Their various branches.

§ 2. Hence, Mathematics, in their general acceptation, embrace as many branches as there are species of quantities taken into consideration, and these various branches are also distinguished by appropriate denominations, as Geometry, Hydraulics, Optics, &c.

Algebra: its object.

§ 3. Algebra considers quantities in an abstract manner; that is, it considers in quantities those properties and relations which are common to all the various species; and we may add: That which Logic is to mental philosophy and mental sciences of every description, in some measure Algebra is to the mathematical sciences.

Generality of Algebraic questions. Algebraic symbols.

§ 4. Algebraic questions are consequently quite general, as well as the symbols used to represent the quantities. These symbols are commonly the letters of the Latin and Greek alphabet.

Algebraic questions connected with arithmetical questions. Relation of magnitude.

§ 5. Algebraic questions and operations are, besides, strictly connected with numerical or arithmetical questions. Because, whenever one quantity is compared with another of the same kind, for instance, weight with weight, space with space, &c., the relation is no other than numerical. This relation is a relation of magnitude.

Relation of opposition.

§ 6. Another relation, we may say of opposition, depends on the different manner of the existence of quantities. This opposition is designated by the denominations of positive and negative quantities. So, for example, two forces acting in the same straight line, but in a direction opposite to each other, if compared, are respectively positive and negative.

Signs.

§ 7. When a quantity, for instance, a, is destined to represent a positive one, the sign $+$ (*plus*) is frequently placed before it. When the quantity is negative, the sign $-$ (*minus*) is always prefixed to the symbol.

Use of the signs.

§ 8. When, therefore, in the same question we meet with the quantities $+a-b$, it is always understood that a is positive with reference to b, and b is negative with reference to a. And, *vice versa*, if two quantities are given opposite to each other, and before the first we put no sign or the positive sign, the negative sign is then to be constantly put before the second.

How opposite quantities are mutually influenced.

§ 9. When quantities of different signs, suppose two, are collectively taken, their value is then equivalent to a third quantity, which is the difference of the absolute value of them, and whose sign is either positive or negative according as the greater absolute value of the two quantities is that of the positive, or that of the negative. For example: two forces, $+B$ and $-b$, if applied to the same material point and along the same straight line, their effect is the result of their simultaneous and collective action. But, if we suppose B to impel the point twice as much as b, or, which is the same, the absolute value of B to be twice as great as that of b, since the forces act in opposition, the effect of b will be counteracted by that of B; and one-half of B (which is equal to b) will produce alone the effect in the positive direction of B. That is, the collective action of the two forces is equal to that of their difference, and this difference acts in the direction of the greater force.

Coefficient.

§ 10. When a symbol, for example, b, is adopted to represent a certain quantity, and it happens that in the same investigation another quantity occurs whose magnitude is twice, three times, &c. the magnitude of the former b; instead of making use of another symbol, or repeating the same, we write only once the symbol b, placing before it a figure to show how many times the quantity is taken. This number is called *coefficient*, which means *making together* with the symbol, the whole of a quantity. If, for instance, the quantity B is three times as great as b, or C five times as great as c, instead of writing B and C, we would write $3b$ and $5c$.

Similar terms.

When two or more terms differ only in the coefficient, they are termed *similar*. For example, $5b$, $2b$, or $3c$, $7c$, $12c$, are similar terms.

Sign of addition.

§ 11. Let us remark here also, than when a quantity is to be added to another quantity, b for instance to a, or several quantities are to be added to another, this addition is commonly expressed by interposing between the quantities or *terms* the positive sign $+$, which, for this reason, is termed also a sign of addition. Suppose, for example, that the quantities b, c, d are to be added to a, this will be indicated by writing, $a + b + c + d$.

Sign of subtraction.

§ 12. When, on the contrary, a quantity is to be subtracted from another, the quantity from which we subtract is first written, then the other, and the negative sign is placed between them. If, for instance, b is to be subtracted from a, we will write $a - b$.

Equations and inequalities.

§ 13. Comparing together quantities of the same kind, for example, weights with weights, surfaces with surfaces, &c., we will find them either equal or not. Suppose now, for the sake of simplicity, only two such quantities, which we will call a, b. If they are equal, then we write $a = b$, and the sign $(=)$ of equality is read *equal to;* the terms so compared, considered as forming a single expression, are called an *equation.* If the same terms represent two unequal quantities, then a is either greater or less than b; in the former case the inequality is expressed by writing $a > b$, in the second, $a < b$; that is, we place between the terms the angle, or sign of inequality, with the vertex towards the less of the unequal terms.

Monomials and polynomials.

§ 14. Any algebraical expression whose symbols are not separated by positive or negative signs, or the signs of equality or inequality, is called term or *monomial.* For example, the symbol b, together with the coefficient 5, constitute the monomial $5b$. When two terms are separated by a positive or negative sign, the expression is then called *binomial;* if three such terms are separated in the same manner, they form then a *trinomial,* &c.; and in general these expressions are called *polynomials.*

Members of equations.

§ 15. When two or more terms are separated by the sign of equality or inequality, these terms constitute the *members* of the equation or inequality. For example, in the equation, $a + b + c = m - n$, the trinomial $a + b + c$ forms the first member, and the binomial $m - n$ the second member of the equation. Likewise, in the inequality $p + q > f - d$, the first binomial is the first member, and the second binomial forms the second member of the inequality.

Constant and variable quantities. Functions.

§ 16. Any algebraical expression whose value depends on the value of a *variable* quantity, is called function of that variable. For instance, the monomial $6x$ depends on the value given to x. So, likewise, in the equation, $y = a + x$, supposing a an invariable or *constant* quantity, and x variable, y necessarily depends on the value of x; hence, y is a function of x. When such quantities as y are functions of another, or other quantities, this is expressed by the index f, and, instead of writing, for example, y or $a + x$, we simply write, $f(x)$, or $y = f(x)$. The index f must be varied when different functions occur in the same question, and we then make use of F or φ, or some other letter.

When, therefore, a quantity a, or several quantities $a + b$, &c., are submitted to any operation, the result is a function of those quantities, because it depends on the same quantities; so that, if instead of a or $a + b$, we should submit to the same operations other quantities, for instance, A and A + B, the result would necessarily be different. But if two or more quantities are equal among themselves, and are submitted to the same operations or equally modified, the result must necessarily be the same. Hence, if a is equal to b and c, &c., and a is submitted to such an operation as to give for result, $f(a)$, if we submit b and c to the same operation, with the equation,

$$a = b = c = \ldots.$$

we must have also,

$$f(a) = f(b) = f(c) = \ldots.$$

That is to say, *the members of an equation equally modified form another equation.*

This deduction cannot for the present be developed nor illustrated by examples, but its frequent application will soon supply copious illustrations.

Modification of quantities; their mutual relations.

§ 17. Quantities are essentially capable of increase and diminution; and considering any quantity in an abstract manner, we cannot conceive any other modification of it, except that which is performed by addition or by subtraction, or by equivalent operations. Again, quantities may be compared together, either by a simple or complex comparison. This is all that concerns quantities, generally considered; hence, Algebra may be conveniently divided into two parts: the first of which has for its object operations on quantities; the second, to investigate and discover the properties, connections, and dependences of quantities, according to their various comparisons and combinations.

Division of the treatise.

FIRST PART.

ALGEBRAIC OPERATIONS.

CHAPTER I.

DEFINITIONS AND OPERATIONS ON MONOMIALS.

ARTICLE I.

Addition and Subtraction.

Algebraic and arithmetical additions compared.

§ 18. Addition.—Numerical or arithmetical addition consists in finding out a number containing in itself as many units and fractions of units as there are in all the numbers to be added together. For example, to find out the number 12, which contains in itself as many units as there are in the numbers 2, 4, 6, is to make the *sum* or addition of these numbers, and this sum is expressed (11, 13) by $2 + 4 + 6 = 12$. But, with regard to algebraical quantities; for instance, $a, b, c, \ldots.,$ although the sum is represented as in numbers, namely, $a + b + c + \ldots,$ yet, on account of the more general signification of the algebraic symbols, the operation is not equally simple as for numbers. The quantities represented by algebraic symbols have, indeed, a numerical value; nay, this value is the one taken into account in addition, as well as in other algebraic operations. But quantities may have either a positive or a negative value; so that a, for example, may be negative with regard to b and c. Then the numerical and relative value of a is to be expressed, for instance, by -3, while the others are expressed, suppose by $+4$ and $+5$: in this sum or collection the negative part is destroyed by the positive, and since $5 + 4 = +9$ and $9 - 3 = +6$, a quantity m, whose numerical value is $+6$, will represent the sum of the given quantities a, b, c, and we will have

$$a + b + c = m.$$

From these remarks we deduce the following definition, and two practical rules:

Definition of algebraic addition.

The sum of a number of quantities is a monomial, whose numerical value is the excess of the numerical value of the quantities affected by one sign, over the numerical value of the quantities affected by the opposite sign, and the sign of which is the sign of the same excess.

Terms all affected by the same sign.

Some consequences easily derived from this definition will make it more clear. First, if all the quantities to be added have either a positive or a negative sign or value, the sum has, likewise, a positive or negative value, and the numerical value of this sum contains as many units and fractions as there are in all the numerical values of the quantities, added exactly as for numbers. Secondly, if the numerical value of the quantities to be added amount to the same for those which have a positive, and for those which have a negative value, the sum is then equal to zero. A third consequence needs not to be pointed out, since it obviously appears from the definition itself.

Equal numerical value of quantities affected by opposite signs.

Rules.

RULE 1.—*When the quantities to be added are merely represented by symbols, we consider them as having a positive sign, and their sum is expressed by writing in succession the same quantities, and placing the positive sign between them.*

For example, the sum of the quantities

$$a, b, c, d, \ldots.$$

is expressed by $\quad a + b + c + d + \ldots.$

RULE 2.—*When the sign is placed before the quantities to* *be added, then the sum is represented by writing, likewise, the symbols in succession, each with its own sign; but the sign of the first is not written unless it be negative.*

For example, the sum of the quantities,

$$a,\ b,\ c,\ -g\ -h\ -k,$$

is expressed by $\quad a+b+c-g-h-k.$

It is evident that a sum will be always equal to a certain unvariable monomial expression, whatever be the order in which the terms are written. So, for instance, calling m the equivalent monomial expression, we may write

$$a+b+c-g-h-k=m,$$

or $$a-g+b-h+c-k=m, \text{ \&c.}$$

The proposed examples are the most general. In more particular cases there occur simplifications or reductions of terms, which we will soon see in other examples.

Doctrine of signs. § 19. We have already remarked, that the sign placed before a symbol is not always the same sign as that of the numerical value of the quantity represented by it; and although, generally, the quantity a or $+a$ is considered as having a positive numerical value, and the quanity $-b$, a negative numerical value, it happens, however, in mathematical investigations, that the numerical value of a positive quantity is sometimes found negative, and *vice versa.* Hence, some questions arise concerning the final sign to be given to a quantity, which deserve to be noticed here. And, to give to our researches a quite general character, let us first remark, that one or more signs by which an algebraical symbol can be occasionally affected necessarily affect the numerical value itself, and *vice versa;* secondly, an algebraical symbol is frequently a symbol of another, or other symbols, sometimes affected by the same, sometimes by the opposite signs. For instance, we may have

$$a=+\mathrm{A}$$

$$\text{or } a=-\mathrm{A};$$

hence, $$+a=+(+\mathrm{A}), \text{ or } +a=+(-\mathrm{A})$$

$$-a=-(+\mathrm{A}), \text{ or } -a=-(-\mathrm{A}).$$

Product of signs. The expressions of these two sets are manifestly opposite, but $+(+\mathrm{A})$ is equivalent to $+\mathrm{A}$; hence, [illegible]$(+\mathrm{A})$ must be equivalent to $-\mathrm{A}$; again, $+(-\mathrm{A})$ is equivalent to $-A$; hence, $-(-A)$ must be equivalent to $+A$. Therefore, we derive this general inference:

A double sign placed before an algebraical symbol, is equivalent to the positive sign, when the two are either both positive or both negative; it is equivalent to the negative sign, when one of the two is positive and the other negative.

But, suppose A to be a symbol of another algebraical symbol; for example, $+a$ or $-a$, we will have

$$++A=++(+a),$$

or

$$++A=++(-a),$$

and, continuing the same process, we see that any number of signs may affect an algebraical symbol, but the same signs may be easily reduced to a single one. When, for example, b is affected by a number of signs, as follows:

$$+--+-b;$$

make first, $+-b=c$, we will have

$$+--+-b=+--c;$$

make again, $--c=d$, we will have

$$+--+-b=+d=d;$$

but $+-b=-b$; hence, $c=-b$,
and $--c=+c$; hence, $c=d$,
and consequently, $d=-b$;
but $d=+--+-b$;
hence, $+--+-b=-b$;

and in general, when the original number of signs contains an even number of negative ones, the final sign is always positive; and when the original number of signs contains an odd number of negative, the final sign is always negative. In fact, no sign is changed from positive into negative, and *vice versa*, except by the influence of a preceding negative sign; hence, the first negative sign determines a negative sign for the symbol, the second changes it into positive, the third into negative, &c.

The analogy between the mutual influence of signs when applied to the same quantity, and the influence of terms affected by different [illegible]ns when multiplied together, has given to the final sign in question the name of product of signs, although this result is altogether different from that of multiplication.

Examples and problems.

§ 20. Let $-2b)$ $+m)$ $+3b)$ $-3m)$ $+f)$ $+2m)$ be terms to be added. In this exam[illegible] the similar terms, $-2b$, $+3b$, and $+m$, $-3m$, $+2m$ may be reduced to a less number, because (9) $3b-2b$ is evidently equal to $+b$ or b, and $m+2m=3m$; hence, $m+2m-3m=0$; therefore, the sum of all the terms is given by $b+f$, that is, $-2b+m+3b-3m+f+2m=b+f$.

Let the terms given for another addition be $16m) + 12c) - 4r) + s) - 12m) - 13c) - 3s) + r) + c) - m) + 7r) - 13s)$. Select first the similar terms, and dispose them as follows:

$$\begin{array}{lrrrr}
 & +16m & +12c & -4r & +\ s \\
 & -12m & -13c & +\ r & -\ 3s \\
 & \underline{-\ \ m} & \underline{+\ \ c} & \underline{+7r} & \underline{-13s} \\
\text{Sum} & +\ 3m & 0 & +4r & -15s.
\end{array}$$

And since the collection of the separtial sums gives the total sum, we will have

$$16m+12c-4r+s-12m-13c-3s+r+c$$
$$-m+7r-13s=3m+4r-15s.$$

From these examples it is plain, that the addition of simple monomials consists in a bare reduction of similar terms, and this reduction is performed by taking the sum of their coefficients when they are affected with the same sign, or by taking the difference of the same coefficients when affected with opposite signs. Let us now propose some problems to be resolved with simple addition of monomials.

First problem.

Twelve divisions of soldiers, containing each $2n$ soldiers, are in a castle when the enemy commences the attack; 2 of these divisions take to flight during the assault; 4½ divisions perish in the conflict. The assailants gain the battle, and their general with 8 divisions, each containing r men, enters into the fort when it is still occupied by the defenders.

We ask what is the amount x of combatants in the fortress after the entrance of the victorious general?

This problem, besides giving another example of addition, gives also occasion to exemplify still more the relative signification of positive and negative terms. The quantity here inquired is a number of men; considering, therefore, as positive all that tend to add to this number, we must consider as negative all that tend to diminish the same number, being evidently quantities opposite to one another. Hence, the terms given by the problem, will be as follows:

Twelve divisions of men, each containing $2n$ soldiers, give the term, $+12.2n$, or $+24n$.

Two divisions of men leaving the fort, give the term $-2.2n$, or $-4n$.

Four and a half divisions lost in the battle, give the terms, $-4.2n, -n$; that is, $-8n, -n$.

The general entering into the castle, gives the term, $+1$.

And the eight divisions containing each r soldiers, give the term, $+8r$. Hence, we have for the required amount

$$24n-4n-8n-n+1+8r;$$

which gives $$x=11n+1+8r.$$

Letters used for unknown quantities.

We may here remark, that x, as well as y and z, and some of the other last letters of the alphabet, are usually employed to represent the quantities to be found, or generally unknown quantities.

Numerical applications of the problem.

But to render the case more determined, suppose $n=50$ and $r=80$, we will have from $x=11n+1+8r$,

$$x=1191.$$

If, instead of $n=50$ and $r=80$, we take $n=80$ and $r=60$, then we have $$x=1361.$$

General character of algebraical questions.

And so we could resolve any number of cases by substituting other values for r and n. And from this the learner may appreciate the general character of algebraical questions.

Problem 2.

Four hunters agree to meet together at the verge of a river after hunting. The first of them shoots

n birds, the second, twice the same number, and $2r$ birds besides; the third shoots as many birds as the first and second killed together; the fourth does not shoot any, but seeing the good success of his companions, takes the birds brought by the first, and one-half of those brought by the second, and throws them into the river.

How many birds remained after this? and how many birds were brought by the hunters?

Ans. to the first question:

$$x = 4n + 3r.$$

Ans. to the second question:

$$y = 6n + 4r.$$

Suppose $n = 10$, $r = 9$, then

$$x = 67, \ y = 96.$$

Suppose $n = 9$, $r = 10$, then

$$x = 66, \ y = 94, \text{ \&c.}$$

Difference.

§ 21. SUBTRACTION.—To subtract a quantity b from another quantity a, means, to find the *difference* between the two quantities; and this difference, which we will call d, is such, that if added to the quantity b, the sum will be a. Hence, we may briefly give the following definition:

Definition.

To subtract b *from* a, *means to find out a quantity* d, *which added to* b *gives* a. Namely,

$$d + b = a.$$

Now let us add the term $-b$ to both members of this equation, we will have $d + b - b = a - b$, that is

$$d = a - b.$$

Suppose, again, b equal first to $+g$, and secondly to $-g$; we will have, in the two cases,

$$-b = -(+g)$$
$$-b = -(-g).$$

Result of signs.

But in the second case, the value of b is opposite to that of the same b in the first case, consequently, since $-(+g) = -(g)$, or since in the first case $-b = -g$, in the second we will have $-b = +g$; that is, $-(-g) = +g$, which exactly corresponds to the doctrine of signs (19); hence, with $b = +g$, we have

$$d = a - b = a - g;$$

and with $b = -g$, we have

$$d = a - b = a + g.$$

In the first case we obtain the difference d, by adding to a a quantity opposite to $+g$; in the second case the required difference is obtained by adding to a a quantity opposite to $-g$; hence, follows this general rule:

Rule for subtraction.

The quantity b *is subtracted from* a, *by adding to* a *a quantity opposite to* b.

General examples.

Thus, for example, $-k$ is subtracted from h, by simply writing

$$h + k,$$

and m is subtracted from n by writing

$$n - m.$$

These are the most general examples of algebraical subtraction of monomials. We will soon propose other examples and problems, in which the difference can be expressed by a single term.

Criterion of magnitude.

§ 22. Let us here observe, that when the difference $n - m$ is positive, then n is said to be greater than m; when the same difference is negative, then n is said to be less than m. The difference, therefore, between two quantities, is the criterion of their relative magnitude; and since by substituting for n any positive number or numerical value, and for m any negative number or numerical value, the difference is certainly always positive, so it follows that any positive number or quantity relatively considered is greater than all negative ones.

Any positive number greater than all negative.

Again, substituting for n and m negative numerical values, but the absolute value of n less than the absolute value of m, the difference is, likewise, always positive; therefore, the greater of two negative quantities or numbers, relatively considered to a certain term, is that

Twofold acceptation of numbers.

which has a less numerical value. We may illustrate the same doctrine as follows: observing first, that numbers are either considered as terms of comparison, or as symbols by which one or more existing objects may be designated. When we consider numbers under this last point of view, the only cipher zero, which excludes all numerical signification, is and signifies nothing. When we consider numbers as terms of comparison, zero is a term to which we may refer all the others as to any number. So it is evident that to say three units above zero, or two units below five, conveys the same conception of the number three. Nay, the term zero is the central term between the ascending series of positive, and the descending series of negative numbers. This being admitted, we observe, moreover, that with regard to positive numbers, all agree that the greater among them is that which is farther from zero, the term below all positive numbers; but zero is in an equal manner above all negative numbers, and the more above, the more they increase in absolute value; referring, therefore, to the same term, zero, negative numbers, we infer that among negative numbers relatively considered those are less that have a greater numerical value.

The term zero taken as mere term of comparison.

Examples and problems.

§ 23. From $5b$ subtract $4b$; we will have the difference

$$5b - 4b = b.$$

From $4b$ subtract $5b$;
we will obtain $4b - 5b = -b.$

From $5b$ subtract $-4b$;
we will have $5b + 4b = 9b.$

From $-5b$ subtract $-4b$;
we will have $-5b + 4b = -b.$

From $-4b$ subtract $-5b$;
we will have $-4b + 5b = b.$

Problem 1.

Ten men pull, with a rope, a heavy stone in a straight line from A towards B, and with a force $10n$. Seven more men pull the same stone in an opposite direction, namely, from B towards A, with a force $7n$. What is the difference x of the action of these two forces?

Ans. It is plain that, considering the action which moves the weight towards B as positive, the opposite action must be negative. Hence, the two terms in question are $+10n - 7n$. Now the second is to be subtracted from the first, therefore,

$$x = 10n + 7n = 17n.$$

Numerical applications.

Suppose $n = 10$ pounds, we will have

$$x = 170 \,..\, p.$$

Suppose $n = 15$ pounds, we will have

$$x = 255 \ldots p.$$

Problem 2.

Four workmen cut each n pieces of timber, and three boys cut each r pieces. What is the difference between the two numbers?

Ans. $$x = 4n - 3r.$$

Numerical applications.

Suppose $n = 50$, $r = 30$, then

$$x = 110.$$

Suppose $n = 90$, $r = 40$, then

$$x = 240.$$

Let us observe, that when we merely intend to take the difference between two numbers affected by the same sign, we only attend to the numerical value.

ARTICLE II.

Multiplication and Division.

In what multiplication consists: nomenclature.

§ 24. MULTIPLICATION.—To multiply a monomial a by another monomial b, means to find a quantity p, whose numerical value is equal to the product of the numerical values of a and b. The monomial a is termed *multiplicand*, b *multiplier*, and both, *factors*. The

Various manners of representing the product.

quantity p is termed *product*, and this product is represented by the factors in any of the following manners:

$$a \,.\, b,\ a \times b,\ ab,$$

and each one of these expressions is read a multiplied by b, or simply ab.

Definition of numerical multiplication.

§ 25. The definition and description of numerical multiplication is frequently given as follows: Multiplication is the addition of the multiplicand repeated as many times as there are units in the multiplier. This definition (when we merely consider the absolute value of the product) is correct so far as the multiplier is a whole number; but when it becomes a fractional one, that is, when the multiplier is a fraction of unity or even contains some units, but a fraction of unity besides, the given definition cannot then be rigorously applied. A definition which comprehends the object in its full extension, supposing, namely, the multiplicand A and the multiplier B to be any two numbers, is the following: To multiply A by B, is *to derive from* A *through addition a number in the same manner as, through the addition of the same element, the number* B *is derived from positive unity.* That is, the operation to which positive unity must be submitted in order to give through addition the number B, is the same operation to which A must be submitted to obtain the product of the numbers A and B. Now, B represents a rational number, (either whole or fractional,) or an irrational one. Let us examine each of these cases, and we will have a complete explanation of the last definition.

Case of the multplier. Whole number and positive.

Suppose, first, B a whole and positive number. The simple addition of unity repeated as many times as there are units in B, is the operation to be made about unity to derive from it B through addition. The multiplication, therefore, of A by B, consists in this case in making the addition of A taken as many times as there are units in B, which accords exactly with the first definition. From this we derive a consequence concerning the sign which affects the product; a consequence applicable also to the cases to be considered hereafter.

Consequence concerning the sign of the product.

Since positive unity, taken as it is, forms by repeated addition the positive multiplier B, so A, taken as it is given, and repeatedly added to itself, gives the product

The multiplier negative.

of A by B. Hence it follows, that when A also is positive, the product is positive. But when the multiplier A is given negative, and B is still positive, then the product, being a sum of negative terms, is necessarily negative. Suppose now, B a whole negative number, then B cannot be immediately obtained from positive unity, but we must first change its sign. But according to the definition, to obtain the product of A by B, we ought to operate about A as about positive unity to obtain B. So in the case of B negative, the sign of the multiplicand A is to be changed; then observing how many units are in B, add A to itself, as in the preceding case, but with the sign changed, which, consequently, is the sign of the product. Therefore, when B is negative, and A also negative, the product is positive; when B is negative but A positive, the product then is negative. Hence, the known rule, like signs give a positive product; unlike signs, negative.

Case of the multiplier fractional.

When B is a fractional number, having, for example, n for its numerator, and m for denominator, in this case, to obtain B from unity, we must take, first, one m^{th} part of unity and add it n times to itself, because in this way only, through the addition of the same element, we can derive B from unity. Operating now upon A in like manner, we will have first $\frac{A}{m}$, which represents the m^{th} part of A; taking then n times this element, which is expressed by placing the coefficint n before $\frac{A}{m}$, we will have the product $n\frac{A}{m}$, corresponding to the factors A, $\frac{n}{m}$.

Case of the irrational factors.

In one of the following paragraphs we will dwell on irrational numbers. For the present it is enough to observe that they cannot be expressed like rational numbers, although we may conceive a series of rationals, continually and indefinitely approaching to any irrational. Hence, whenever an irrational number is to be used for any purpose, we must necessarily make use of a rational near it. Therefore, in the case of irrational numbers, the multiplication will be performed with rational numbers, and, consequently, the foregoing remarks are applicable to this case also.

Arithmetical rules applicable to quantities.

§ 26. Considering the numerical value and sign of quantities, it is plain that the same arithmetical rules are to be followed with regard to quantities for that which concerns the form and sign of the product. The rule of signs may be derived indeed from the definition. But since all agree in admitting that $+a$, multiplied by $+b$, gives a positive product, we may infer the same rule as follows:

Mutual influence of factors.

The factors are mutually influenced in effecting the product, and this influence is twofold: the one numerical, or of magnitude; the other of sign. Suppose, now, the numerical value of the factors $+a$, $+b$ to remain unvaried, and change the sign of either of them; this change must necessarily affect the product $+p$, and this cannot be done except by the change of the sign of the same product, and so admitting

$$+a \times +b = +p,$$

we must admit, also, the two following equations:

$$+a \times -b = -p;$$
$$-a \times +b = -p.$$

Take again either of these two equations, for instance, the last, and change the sign of b; this will again produce an equivalent change in the product, and we will have

$$-a \times -b = +p.$$

Treating of the multiplication of polynomials, we will come to the same consequence by another process; meanwhile we may infer the general rule.

Rule of signs.

The sign of the product is positive when both factors are affected with the same sign: it is negative when the factors are affected with opposite signs.

In the practical application of this rule we usually say, plus by plus, or minus by minus, give plus; plus by minus, or minus by plus, give minus.

Various forms of the numerical value of quantities.

§ 27. Thus far we have considered the factors in their most general acceptation, and only two. But the numerical value, which is the one taken into account, specifies in some measure the quantities, because this value is either whole or fractional; hence, four cases can take place with regard to the factors a and b. We may first suppose the numerical values of both of them to be whole numbers; secondly, both fractional; thirdly, the numerical value of the multiplicand a whole number, and that of the multiplier fractional, and we may finally suppose the multiplier a whole number, and the multiplicand fractional.

The student being familiar with the numerical operations, it is not necessary here to dwell upon them: it will be profitable, however, to place before his eyes the general formulas of those which concern our present question, leaving, if necessary, to the teacher the care of making numerical substitutions.

General formulas of numerical multiplication.

Suppose m, n, k, h to represent whole numbers, and $\frac{n}{m}$, $\frac{h}{k}$, fractional ones. With them we may represent the above-mentioned cases; and calling p the product of m by n, we will have

$$(f)\quad\begin{cases} m\,.\,n = p \\ \frac{n}{m}\,.\,\frac{h}{k} = \frac{n\,.\,h}{m\,.\,k} \text{ and } \frac{h}{k}\,.\,\frac{n}{m} = \frac{h\,.\,n}{k\,.\,m} \\ m\,.\,\frac{h}{k} = \frac{m\,.\,h}{k} \text{ and } \frac{h}{k}\,.\,m = \frac{h\,.\,m}{k} \\ \frac{n}{m}\,.\,h = \frac{n\,.\,h}{m} \text{ and } h\,.\,\frac{n}{m} = \frac{h\,.\,n}{m}. \end{cases}$$

From these formulas we will soon derive a general and useful consequence.

How the arithmetical rules of multiplication follow from the definition.

§ 28. Let us observe, meanwhile, that the arithmetical rules expressed by the (f) cannot be arbitrary, and if right, they must necessarily follow from the definition of multiplication. Examining (25) the case of a fractional multiplier, we have touched this subject, which we will develop here. And, first, suppose $\frac{h}{k}$ to be the multiplicand, and $\frac{m}{n}$ the multiplier, which indicates that the n^{th} part of unity has been taken m times. Hence, to obtain the product of $\frac{h}{k}$ by $\frac{m}{n}$, according to the definition

(25), the n^{th} part of $\frac{h}{k}$ is first to be determined, and the same is then to be taken m times. Now, the n^{th} part of $\frac{h}{k}$ is $\frac{h}{k.n}$. Suppose, in fact, a straight line divided into h equal parts, these parts may represent the h of the fraction $\frac{h}{k}$; suppose, besides, each one of these parts to be subdivided into n equal parts, (which is the same as dividing unity into $n.k$ parts,) the h part of our line will then become $h.n$, but each one of these is equal to the $(k.n)^{th}$ part of unity; therefore, the $h.n$ parts of the line will represent the fraction $\frac{h.n}{k.n}$, but the same line represents also the fraction $\frac{h}{k}$; therefore, $\frac{h}{k} = \frac{h.n}{k.n}$. Compare now together, the fractions $\frac{h}{k.n}$ and $\frac{h.n}{k.n}$; the first is n times less than the second, but the second is equal to $\frac{h}{k}$; therefore, $\frac{h}{k.n}$ is n times less than $\frac{h}{k}$, or which is the same, $\frac{h}{k.n}$ is the n^{th} part of $\frac{h}{k}$. To complete now the multiplication of $\frac{h}{k}$ by $\frac{m}{n}$, the n^{th} part of the first fraction is to be taken m times, which is evidently obtained by multiplying by m the numerator h of the fraction $\frac{h}{k.n}$. Hence, $\frac{h}{k} . \frac{m}{n} = \frac{hm}{kn}$, exactly as the rule prescribes; the product of one fraction multiplied by another is equal to another fraction whose numerator is the product of the numerators of the factors, and the denominator is the product of the denominators of the same factors.

Mechanical artifice, showing the aliquot part of a fraction.

Let us now come to the cases of the multiplicand whole number, and of the multiplier fractional, and *vice versa.* In the first of these two cases, reasoning as above, we will have $m . \frac{h}{k} = \frac{m.h}{k}$; and in the second, it is plain that $\frac{n}{m} . h = \frac{n.h}{m}$. That is, the product of a whole number by a fraction, and *vice versa,* is obtained by multiplying the numerator of the fraction by the other factor.

The multiplicand may become multiplier, and *vice versa,* without affecting the product.

§ 29. The product p of the first formula (f) is given by m units, repeated n times. But to add m units n times, is the same as to add n units m times. We can see it by making use of a mechanical artifice. Range on a horizontal line a row of m dots, and from each dot draw a vertical line; range again on each one of these verticals n dots, commencing with that already marked on the horizontal line. In this manner we have m dots repeated n times, and consequently the whole number of dots is the product of m by n. But since, on each vertical line there are n dots, and these lines are m in number, we have also n dots repeated m times; that is, the product of n by m given by the same number of dots; hence, we may always write

Case of the factors: whole numbers.

$$m \, . \, n = n \, . \, m.$$

Other cases.

Therefore we may write also $n \, . \, h = h \, . \, n$, and $m \, . \, k = k \, . \, m$, and consequently $\frac{n \, . \, h}{m \, . \, k} = \frac{h \, . \, n}{k \, . \, m}$, but from the second of (f), we have $\frac{n \, . \, h}{m \, . \, k} = \frac{n}{m} \, . \, \frac{h}{k}$, and $\frac{h \, . \, n}{k \, . \, m} = \frac{h}{k} \, . \, \frac{n}{m}$; therefore,

$$\frac{n}{m} \, . \, \frac{h}{k} = \frac{h}{k} \, . \, \frac{n}{m}.$$

Reasoning in the same manner, we deduce from the remaining formulas (f) that

$$m \, . \, \frac{h}{k} = \frac{h}{k} \, . \, m.$$

General inference for algebraical factors.

Whatever be, therefore, the numerical value of the algebraical terms a and b, we generally infer that

$$a \, . \, b = b \, . \, a.$$

The product of any number of terms is the same, whatever be the order in which the factors are disposed.

§ 30. The same inference may be applied to any number of algebraical terms $a, b, c, d, e \ldots$

Observe first, that if any number of quantities, having all a whole numerical value, bring the same product whatever be the order in which they are taken; any number of algebraical quantities will bring the same product also when their numerical values are fractional or partly fractional, and partly whole numbers, whatever be the order in which they are taken, since the operation is always performed upon the whole numbers of numerators and denominators. It is enough, therefore, to demonstrate here, that whatsoever be the order in which whole numbers, or quantities having whole numerical values, are taken, their product will be always the same. Let three such quantities be a, b, c. To multiply a by b, is to take a as many times as there are units in the numerical value of b; that is, $a + a + a + a +$. Again, to multiply this product by c is to add the whole series of terms $a + a + a + a$ repeated as many times as there are units in c. Now, b terms repeated c times give a number of them equal to the product $b \times c$. To multiply, therefore, the product $a \,.\, b$ by c, or a by the product $b \,.\, c$, gives the same result; hence, generally,

Three factors.

$$a \,.\, b \times c = a \times b \,.\, c;$$

and since $a \,.\, b = b \,.\, a$, $b \,.\, c = c \,.\, b$, so we will have also,

$$b \,.\, a \times c = a \times c \,.\, b = a \,.\, c \times b,$$

and, in like manner,

$$a \,.\, c \times b = c \,.\, a \times b = c \times a \,.\, b,$$

and so on. So that we may evidently infer that three factors multiplied in any order whatsoever, give constantly the same product.

Four factors.

Add now another factor, and make

$$a.b.c.d = \mathrm{P}.$$

The first three may be changed at pleasure, and the factor d will always multiply the same quantity; but calling p the partial product of the first two factors, the same product P can be represented also by $p.c.d$, or by $p.d.c$; that is, resolving again p into its factors,

$$a.b.d.c = \mathrm{P}.$$

But again, whatever be the order in which a, b, d are taken, their product will remain unvaried; the factor d, therefore, which was the last, can become the third, the second, and the first, while the other three factors may be arranged from the beginning in any manner whatsoever; but this evidently embraces all the possible cases of combinations of the four factors; therefore, the product P made by four factors will be always the same, whatever might be the order in which the factors are taken. We may reason in the same manner when the factors become five, because the first four may be changed at pleasure; considering then the first three as a single term, the place of the fourth may be changed with that of the fifth, which, together with the three preceeding, will always give the + . . . same product, whatever be the manner in which it is combined with the others; the same consequence, therefore, can be inferred with regard to five as with regard to four factors, and the same with regard to six, with regard to seven, and generally with regard to any number *n* of factors.

Sign to be given to the product of several factors.

§ 31. It remains now for us to see what is the sign to be given to the product, when several terms are multiplied. The factors are either all positive or all negative, or partly positive and partly negative; in the first case the product is evidently positive; in the second it is positive likewise, if the number of terms is even, because the first factor with the second make a positive product, which the third changes into another negative, and this, with the fourth factor, makes again another product positive; and so on. If the negative factors are three, their product is negative; if four, positive; if five, negative; and hence, generally, when all the factors are negative, their product is positive, when their number is even; their product is negative, when their number is odd. The same is to be said when only a portion of the factors is negative; that is, when the number of these factors is even, the total product is positive; when the number of negative factors is odd, the total product is negative. In fact, the first negative factor after some positive factors makes the whole product negative, and if other positive factors occur, the successive products will remain still negative; but when another negative factor occurs, then

Three cases.

the product becomes positive, and such will remain until the third negative factor comes; and therefore, the following rule will determine the sign to be given to the total product in all cases:

Rule. *The sign is negative whenever the number of negative factors is odd; otherwise, it is always positive.*

Product of the same factors. § 32. When different factors are given, it may occur that some of the factors are repeated: in this case, instead of writing, for example, $a.a.a$, we write a^3; we write, namely, a only once, and above it the number of times the same quantity is taken as a factor, and this number is called *exponent*. **Exponent.** Of such exponential quantities, and of their reduction, we will speak more fully in its proper place in the next article; however, we cannot o[illegible] here adding a few remarks concerning this subject, inasmuch as it is connected with simple multiplication. And first, if two or more exponential quantities, for example, b^5 and b^4 are to be multiplied together, their product will be represented either by $b^5.b^4$, or by b^9; since the signification of these expressions is the same, that is, the sum 9 of the partial exponents 5, 4, signifies that b is taken as a factor nine times in both cases. Therefore, in cases similar to this, it is enough to write once the quantity, and give to it for exponent the sum of the partial exponents. *Vice versa*, since the number 9 is equal to the sum of 5 and 4, or 6 and 3, &c. We may, for the same reason, write $b^9 = b^5 . b^4 = b^6 . b^3 = \ldots$; and this also can be evidently applied to all similar cases.

Observe also, that, since the order in which the factors are taken does not change the product, (30,) the products $a.a.b.b$ and $a.b.a.b$ are equal to one another. Now, $a.a.b.b = a^2b^2$, and $ab.ab = (ab)^2$; therefore, $a^2b^2 = (ab)^2$, and for the same reason, if any number of factors having the same exponent are to be multiplied together, we may write once the product of the simple quantities, and apply to this product the common exponent; for instance, the product $a^5.b^5.c^5$ is equivalent to $(a.b.c)^5$.

EXAMPLES.

Examples and problems. § 33. Given factors. Product.

(1.) $3a,\ m.n,\ -\frac{1}{6}g,\ -r \ldots\ldots + \frac{1}{2}agrmn.$

(2.) $16m,\ -12n,\ -\frac{1}{4}b,\ \frac{1}{8}c,$
$-gdf,\ \frac{1}{2}h,\ \frac{1}{3}k \ldots\ldots\ldots\ldots -mnbcgdfhk.$

(3.) $4ab,\ -2bc^2,\ -mad,\ d^3 \ldots + 8(abcd)^2m.d^3.$

(4.) $-7abc,\ -3abcd^2,$
$\frac{1}{4}abcde^3,\ -\frac{1}{5}abcdef^4 \ldots\ldots -\frac{21}{20}(abcdef)^4.$

(5.) $abcde,\ -abcd,$
$+abc,\ -ab,\ +a \ldots\ldots + a^5b^4c^3d^2e.$

(6.) $a^2b,\ -bc^3,\ +cd^4,\ -g^5f^5 \ldots + (ab)^2.(cd)^4.(gf)^5.$

(7.) $ga^3bc,\ -b^2c^2,\ +\frac{1}{3}a^2b^3c^3,$
$-4ad^3 + \frac{1}{12}ad \ldots\ldots\ldots + (abcd)^3.(abc)^2.(ab).a.$

(8.) $4a^3,\ -5a^2b,\ -8ab^2,\ 2b^3 \ldots + 320(a.b)^6.$

(9.) $a^2b,\ -ab^2,\ +c,\ -d,$
$+cd,\ -abcd,\ +dc \ldots - (abcd)^4$

(10.) $4am,\ -16bc,\ +\frac{1}{8}m^2,$
$-mn + \frac{1}{8}brf,\ -bcd \ldots - m^4.b^3.c^3.a.n.r.f.d.$

Problem. A general, in order to exercise his soldiers, ranges them on a field before the castle, and divides the whole army in two sections fronting each other, the one under the walls of the castle, the other opposite to it. During the exercise the general rides up and down between the opposite ranks, and when the exercise commences he rides, having the castle at his right hand, and he goes n times up in this manner, and returns n times to his former station. Each time the general rides from his first position to the second, g ranks of v men pass from the left to the right hand of the general, and r ranks, each containing p men, pass from the right to the left hand. When the general returns to his former station, each time f ranks of s men pass from his left to his right hand, and q ranks of t men pass from his right to his left.

We ask, first, how many ranks, and how many men go towards the castle: again, how many ranks and men march over to the other divisions from the castle during the n times that the general goes from his first station to the opposite side of the field?

Ans.

$$\left.\begin{array}{l}\text{Ranks}\ldots n.g\\ \text{Men}\ldots\ldots n.g.v\end{array}\right\}\text{ passing on the castle side.}$$

$$\left.\begin{array}{l}\text{Ranks}\ldots n.r\\ \text{Men}\ldots\ldots n.r.p\end{array}\right\}\text{ going from the castle.}$$

2d. How many ranks and men go to and from the castle the n times the general returns to his former station?

Ans.

$$\left.\begin{array}{l}\text{Ranks}\ldots n.q\\ \text{Men}\ldots\ldots n.q.t\end{array}\right\}\text{ going to the castle.}$$

$$\left.\begin{array}{l}\text{Ranks}\ldots nf\\ \text{Men}\ldots\ldots n.fs\end{array}\right\}\text{ going from the castle.}$$

3d. How many ranks and men go towards the castle, and how many go from the castle during the whole military exercise?

Ans.

$$\left.\begin{array}{l}\text{Ranks}\ldots ng+nq\\ \text{Men}\ldots ngv+nqt\end{array}\right\}\text{ going to the castle.}$$

$$\left.\begin{array}{l}\text{Ranks}\ldots nr+nf\\ \text{Men}\ldots nrp+nfs\end{array}\right\}\text{ going from the castle.}$$

4th. What is the difference between the number of the ranks and men passing to the castle side, and that of the ranks and men passing to the opposite side?

Ans. Ranks... $(ng+nq)-(nr+nf)$.

Men... $(ngv+nqt)-(nrp+nfs)$.

Or, if $(ng-nq)$ and $(ngv+nqt)$ are the less numbers,

Ranks...$(nr+nf)-(ng+nq)$.

Men.....$(nrp+nfs)-(ngv+nqt)$.

Remarks. Observe that, if we consider the passing from the left to the right hand of the general as positive, we must consider as negative the passing from the right to the left. But referring the movement of the ranks to the castle, then when the going towards it is considered as positive, the going from the castle must be considered as negative. Again, considering as positive the going of the general from his former station to the opposite end of the field, we ought to consider as negative his returning to the same station. Taking now the movements of the army first with reference to the castle, and then with regard to the general, and following the rules of signs, we will find in both cases the same resolution of the problem.

Problem 2. A steamboat travels at the rate of n miles per hour. How many miles does the steamboat run over in m days, travelling 16 hours a day?

Ans. $x = 16n.m.$

Suppose $n = 15$, $m = 12$, then

$$x = 2880.$$

Suppose $n = 10$, $m = 20$,

$$x = 3200, \text{\&c.}$$

Definition. § 34. DIVISION.—To divide a quantity a by another quantity b, means to find out such a quantity q, which, if multiplied by b, ought to give a for product; a is called *dividend; b, divisor*, and q, *quotient.* From the given definition it follows, that when the dividend is given, this is considered as the product of two factors, one of which is the divisor, and the object of the division is to find out the other factor.

Algebraical expressions of division. The operation of division is designated as follows:

$$\frac{a}{b}, \text{ or } a:b,$$

and each of these expressions is read a divided by a.

Rule of signs. The rule of signs for division must necessarily be the same as that for multiplication. Suppose, in fact, first a and b both positive, since the quotient q multiplied by a positive quantity must give a positive product, q also, in this case, cannot be but positive, that is,

$$\frac{+a}{+b} = +q.$$

Suppose both dividend and divisor affected by a negative sign; in this case also the quotient must be positive. Because, $-b$ multiplied by q ought to give a negative product, which cannot be obtained unless q is positive. Therefore,

$$\frac{-a}{-b} = +q.$$

Suppose the dividend positive, and the divisor negative, in this case the quotient must be negative, because b, a negative quantity multiplied by q, ought to produce a positive, which cannot be obtained except with q negative; hence,

$$\frac{+a}{-b} = -q.$$

The last case is when b is positive and a negative, and in this case also, the quotient is negative; because b, a positive quantity multiplied by q, ought to give the negative a, which necessarily supposes q negative; hence,

$$\frac{-a}{+b} = -q.$$

We infer, therefore, the general

Rule. *The sign of the quotient is positive when both dividend and divisor are affected by the same sign; the sign of the quotient is negative when the dividend and divisor are affected by different signs.*

Some, for brevity's sake, express this rule common to multiplication and division, as follows:—*Like signs produce plus, and unlike signs, minus.*

Various numerical values. § 35. Observe here again that the numerical values assignable to the algebraical terms employed in division may be either whole numbers or fractional, and consequently, the quotient $\frac{a}{b}$, numerically considered, embraces four cases corresponding to those already considered (27'f) for multiplication, and which we will represent here with corresponding formulas, and according to arithmetical rules:

$$m : n = \frac{m}{n},$$

$$\frac{n}{m} : \frac{h}{k} = \frac{n.k}{m.h},$$

$$m : \frac{h}{k} = \frac{mk}{h},$$

$$\frac{n}{m} : h = \frac{n}{m.h}.$$

These rules for division are founded in the definition. In fact, if we multiply the quotient or the second member of each one of the preceding equations by the corresponding divisor, the product will result equal to the dividend.

$$\frac{m}{n} . n = \frac{m.n}{n} = m . \frac{n}{n} = m,$$

$$\frac{n.k}{m.h} . \frac{h}{k} = \frac{n.k.h}{m.h.k} = \frac{n.k.h}{m.k.h} = \frac{n}{m} . \frac{kh}{kh} = \frac{n}{m},$$

$$\frac{m.k}{h} . \frac{h}{k} = \frac{m.k.h}{h.k} = m . \frac{kh}{k.h} = m,$$

$$\frac{n}{m.h} . h = \frac{n.h}{m.h} = \frac{n}{m} . \frac{h}{h} = \frac{n}{m}.$$

Numerical values of compound monomials.

§ 36. The numerical values of each element of a compound monomial sometimes are given separately. In this case the ultimate reduction for multiplication and division involves some complication. Let us here examine the case of monomials, having the fractional form, and suppose $\frac{a}{b}$ and $\frac{c}{d}$ to be such monomials. Representing by m, n, p, r, s, t, u, v whole numbers, let the numerical value of a be represented by $\frac{m}{n}$, and that of b by $\frac{p}{r}$, that of c by $\frac{s}{t}$, and that of d by $\frac{u}{v}$. And to commence with

Case of multiplication.

the case of multiplication, let $\frac{a}{b}$ be the multiplicand, and $\frac{c}{d}$ the multiplier. Now,

$$\frac{a}{b}=\frac{\frac{m}{n}}{\frac{p}{r}}=\frac{m}{n}:\frac{p}{r}=\frac{m.r}{n.p},$$

$$\frac{c}{d}=\frac{\frac{s}{t}}{\frac{u}{v}}=\frac{s}{t}:\frac{u}{v}=\frac{s.v}{t.u}.$$

Therefore, $$\frac{a}{b}\cdot\frac{c}{d}=\frac{m.r}{n.p}\cdot\frac{s.v}{t.u}=\frac{mrsv}{nptu}.$$

but $\frac{mrsv}{nptu}=\frac{msrv}{ntpu}=\frac{ms}{nt}\cdot\frac{pu}{rv}=\frac{m}{n}\cdot\frac{s}{t}:\frac{p}{r}\cdot\frac{u}{v}=a.c:b.d=\frac{ac}{bd}$;

consequently, $$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd},$$

The product is obtained and represented as for whole numerical values.

whatever be the numerical value of each element a, b, c, d of the fractional expressions, or algebraical factors.

Let us now come to the case of division, and let $\frac{a}{b}$ be the dividend, and $\frac{c}{d}$ the divisor, we have, as above,

Case of division.

$$\frac{a}{b}=\frac{m.r}{n.p},\ \frac{c}{d}=\frac{s.v}{t.u};$$

therefore, $$\frac{a}{b}:\frac{c}{d}=\frac{m.r}{n.p}:\frac{s.v}{t.u}=\frac{mrtu}{npsv};$$

but $\frac{mrtu}{npsv}=\frac{murt}{nvps}=\frac{mu}{nv}:\frac{ps}{rt}=\frac{m}{n}\cdot\frac{u}{v}:\frac{p}{r}\cdot\frac{s}{t}=a.d:b.c=\frac{ad}{bc}$;

The quotient is obtained and expressed as for whole numbers.

hence, $$\frac{a}{b}:\frac{c}{d}=\frac{ad}{bc};$$

as for simple numerical division, whatever be the numerical values of each element of the monomials $\frac{a}{b}$ and $\frac{c}{d}$.

Reduction of results to a simpler form.

§ 37. We must here add some remarks concerning the reduction of the quotient, and in general the reduction of the result of any other operation to a simpler form. And first, any quantity or number multiplied by unity, gives a product equal to itself; for instance, $a \times 1 = a$. Secondly, any quantity divided by itself, gives a quotient equal to unity. Because, the quotient, for example, of $a:a$ must be such, that multiplied by a it gives a, which is none else except unity; hence, it follows, that

$$a.\frac{c}{c} = a;$$

and, consequently, since $\frac{a.c}{b.c} = \frac{a}{b}.\frac{c}{c}$,

and $$\frac{a}{c}:\frac{b}{c} = \frac{a.c}{b.c} = \frac{a}{b}.\frac{c}{c};$$

we will have also, $\frac{a.c}{b.c} = \frac{a}{b}, \frac{a}{c}:\frac{b}{c} = a:b.$

Rule.

That is, *a quantity having the fractional form* $\frac{a}{b}$ *remains unchanged, multiplying or dividing the numerator and denominator by any other quantity.*

Reduction to the same denominator.

Hence, any number of such algebraical quantities $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \ldots$ may be easily reduced to the same denominator like numbers; multiplying, namely, the numerator and denominator of each by the product of all the other denominators. So the above quantities, without being changed, can be expressed as follows:

$$\frac{a.d.f}{b.d.f}, \frac{c.b.f}{d.b.f}, \frac{e.b.d}{f.b.d};$$

having all the same denominator.

Applicable to addition and subtraction.

We may here observe also, that to obtain the sum or difference of algebraical quantities having a fractional form and the same denominator, it is enough to take the sum or difference of the numerators and divide it by the common denominator. Hence, if the quantities $\frac{a}{b}$, $\frac{c}{d}$ are to be added or subtracted from one another, we may first reduce them to the same denominator, taking $\frac{a.d}{b.d}$ for $\frac{a}{b}$, and $\frac{c.b}{d.b}$ for $\frac{c}{d}$, and in the case of addition we will have

$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{cb}{bd}=\frac{ad+cb}{bd};$$

in the case of subtraction,

$$\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{cb}{bd}=\frac{ad-cb}{bd},$$

exactly as for numbers.

Examples and problems. § 38.

	Given.	Answers.
(1.)	$ab : m$,	$\frac{ab}{m}$.
(2.)	$abcm : m$,	abc.
(3.)	$a^2bc : ab$,	ac.
(4.)	$ab^2c^2 : bc^3$,	$\frac{ab}{c}$.
(5.)	$mn : \frac{bcd}{mn}$,	$\frac{m^2n^2}{bcd}$.
(6.)	$\frac{m^2b^2}{ac} : \frac{m^3b^3}{a^2c^2}$,	$\frac{ac}{mb}$.
(7.)	$\frac{3mpq}{4a^2b^3} : \frac{4a^3b}{3m^2p}$,	$\frac{9m^3p^2q}{16a^5b^4}$.
(8.)	$mn : \frac{mn}{bcd}$,	bcd.
(9.)	$-fg : rs$,	$-\frac{fg}{rs}$.
(10.)	$m^2b - \frac{mb^3}{3a}$,	$-\frac{3ma}{b^2}$.
(11.)	$-g^3m^4 : -\frac{4a}{7bdg}$,	$+\frac{7g^4m^4bd}{4a}$.

(12.) $-\frac{4q^5fg}{3a^3rs} : \frac{3a^3sr}{4q^5gf} \qquad -\frac{16q^{10}f^2g^2}{9a^6r^2s^2}.$

(13.) $\frac{4bdr}{7m^2q^2} : -\frac{4mq^3}{14b^4d^3r^2}, \qquad -\frac{2b^5d^4r^3}{m^3q^5}.$

(14.) $-\frac{3b^2r}{7fdc^2qs} : -\frac{md^3s^3}{21r^3bc^3q^3}, \quad +\frac{9b^3cqr^4}{fd^4s^3}.$

(15.) $\frac{7m^2b^3}{9sgr^2f^3} : \frac{21f^3r^2gs}{27b^3m^2}, \qquad \frac{b^6m^4}{f^6r^4g^2s^2}$

Problem 1. A certain number of balls is taken n times from an urn. At the end, the amount of the balls extracted is found to be m. How many balls where taken each time?

Ans. $x = \frac{m}{n}.$

Suppose $m = 50$, $n = 10$,
then $x = 5.$

Suppose $m = 56$, $n = 8$,
then $x = 7$, &c.

Problem 2. Three messengers A, B, C leave at once the same town. A arrives at a distance of n miles, B at a distance of r miles, and C at a distance of s miles, at the end of the same number g of days, travelling each one of them an equal number of miles every day. How many miles did A travel each day? how many B? how many C?

Ans. A travelled $x = \frac{n}{g}$ miles.
B " $x' = \frac{r}{g}$ "
C " $x'' = \frac{s}{g}$ "

Suppose $g = 15$ and $n = 450$,
$m = 630$,
$s = 420$,
then $x = 30$, $x = 42$, $x'' = 28$.

Accents or dashes: their meaning.

When symbols are used to represent similar or analogous quantities, instead of changing the symbol, we make use sometimes of the same symbol with one or more accents above it, as in the preceding example, and such symbols are read *x prime*, *x second*, &c.

ARTICLE III.

Formation of Powers and Extraction of Roots.

What a certain power of a quantity is.

§ 39. POWERS.—Let m be a whole number, and a any algebraic quantity. To raise a to the power m, or to form the m^{th} power of a, means to multiply a by itself m times. In this operation a is termed *root*, m the *exponent* or *degree*, or *index;* the *power*, (which may be called p) is expressed by a^m, and this expression is read a to the m^{th} power, or simply a to m. The operation is the same as for numbers.

Root, exponent, degree, power.

Formation of powers embracing all cases of numerical values. Its description and definition.

§ 40. Numerical relations are in all cases applicable to quantities, since, as we have frequently observed, whenever algebraical quantities are submitted to any operation or comparison, their numerical value is taken into account.

Let l, n, p represent three numbers, and suppose p to be obtained from l through multiplication, as n is derived from unity through addition. Then p is the n^{th} power of l, and the expression of the power being l^n, we have $l^n = p$. To raise, therefore, the root l to the power n, is to find out the number p. Now n, a rational number, and if irrational, represented by a rational one, is either a whole number or a fraction. Suppose, first, n to be a whole number, it is then derived from unity through addition by taking

$$1 + 1 + 1 + 1 + \ldots.$$

n times; therefore, the power p or l^n is given by the product

$$l \times l \times l \times l \times \ldots.$$

where l is taken n times, and this is the case considered in the preceding number.

Case of the index whole number.

Case of the index fractional.

Suppose, now, n to be a fraction having r for numerator, and s for denominator. To derive from unity through addition $\frac{r}{s}$, it is necessary first to divide unity into s parts, and take r times one of these parts, namely,

$$\frac{1}{s}+\frac{1}{s}+\frac{1}{s}+\frac{1}{s}+\ldots$$

To obtain, therefore, p or $l^{\frac{r}{s}}$ it is first necessary to determine a quantity α, which, if multiplied s times by itself would give l in the same manner as $\frac{1}{s}$ added s times to itself gives 1. Then, since to obtain $\frac{r}{s}$ we take $\frac{1}{s}$ r times, so the power $l^{\frac{r}{s}}$ will be equal to

$$\alpha\times\alpha\times\alpha\times\alpha\times\ldots$$

where α is taken r times. And, consequently,

$$l^{\frac{r}{s}} = \alpha^r.$$

Definition.

Hence, we derive the following definition:—*The power of any given quantity* a *is the product of factors equal to the same quantity* a, *or the product of factors equal to such an element, which multiplied by itself a certain number of times produces the given quantity* a. From this definition it follows, that when the index n of the power l^n is either equal to 1, or equal to $\frac{1}{s}$, the power then is merely apparent or nominal.

Apparent powers.

Because, in the first case the root l is not multiplied by itself, but simply taken as it is; and so, likewise, the element α, in the second case is not multiplied by itself. Therefore, according to the given definition, neither of the expressions

$$l^1 = l,\ l^{\frac{1}{s}} = \alpha,$$

Cube and square: nominal powers.

is a power. However, as we term l^3 the third power or *cube* of l, and l^2 the second power or *square* of l; so by analogy we term l^1 the first power of l, and $l^{\frac{1}{s}}$ the $\left(\frac{1}{s}\right)^{\text{th}}$ power of l, which last is so far in reality from being a power of l, that on the contrary l is a power of α, as we have seen, and we will better see hereafter.

Powers of unity.

§ 41. The signification of unity is either collective or simple. In the first case 1 is, like any other root, capable of being raised to any power n, and consequently susceptible of corresponding modifications. In the second case 1^n is again an apparent power, since simple unity is incapable of being raised to any power whatsoever, or in other words, simple unity cannot be affected by any index, either whole number or a fraction. Suppose in fact, the index n a whole number, we will have then

Case of simple unity.

$$1^n = 1 \times 1 \times 1 \times \ldots,$$

but 1, multiplied by simple unity is equal to 1, and, consequently, $1^n = 1$. Suppose n a fraction, we must first find the element α, which multiplied a certain number of times by itself gives 1, but this element cannot be any except simple unity; therefore, α in this case is equal to 1. But in the present supposition, the power 1^n is obtained by repeatedly multiplying α by itself; therefore, in this case also, $1^n = 1$. Simple unity, therefore, remains unchanged when raised to any power.

Collective unity.

But when unity has a collective meaning, then 1^n or more simply 1^2, 1^3, … have by no means the same value and signification as 1. However, even in this case we write $1^2 = 1$, $1^3 = 1$, …. $1^n = 1$, not being able to express these units of various orders with other symbols. But, whenever in mathematical investigations such units of various degrees occur, we take notice of their different meaning, or the exponent is left to indicate the order.

Product of powers: case of the exponents whole numbers applied to the same root.

§ 42. Let now m t represent two whole numbers. We have seen (32) already that the product of $m + t$ factors, equal to a, can be expressed by $a^m . a^t$, as well as by a^{m+t}; nay, whatever be the number of such exponents m, t, r, s, …, we will always evidently have

$$a^m . a^t . a^r . a^s \ldots = a^{m+t+r+s+\ldots}.$$

Fractional exponents.

If the exponents become fractional, (and supposing them reduced to the same denominator,) we will have, likewise,

$$a^{\frac{m}{s}} . a^{\frac{t}{s}} . a^{\frac{r}{s}} \ldots = a^{\frac{m}{s}+\frac{t}{s}+\frac{r}{s}+\ldots}$$

Because, for each factor of the first number of this equation, we must first (40) find out a number which multiplied s times by itself gives a. Let this number be ν. We will have

$$a^{\frac{m}{s}} = \nu^m, \ a^{\frac{t}{s}} = \nu^t, \ a^s = \nu^r, \ldots$$

and, consequently,

$$a^{\frac{m}{s}} . a^{\frac{t}{s}} . a^{\frac{r}{s}} \ldots = \nu^m . \nu^t . \nu^r \ldots = \nu^{m+t+r+\ldots}$$

But also, $a^{\frac{m}{s}+\frac{t}{s}+\frac{r}{s}+\dots} = a^{\frac{m+t+r+\dots}{s}} = \nu^{m+t+r+\dots}$;

therefore, $a^{\frac{m}{s}} . a^{\frac{t}{s}} . a^{\frac{r}{s}} \dots = a^{\frac{m}{s}+\frac{t}{s}+\frac{r}{s}+\dots}$.

General inference and rule.

Hence, we generally infer the following proposition and rule:

The product of any number of powers of the same root, is this very root raised to a power equal to the sum of all the partial exponents, whether whole numbers or fractions.

Product of powers: case of the same exponent a whole number applied to various roots.

§ 43. Let us resume again the whole number m as common exponent of the roots $a, b, c, \dots$ The powers $a^m, b^m, c^m, \dots$ multiplied together, will manifestly give $a^m . b^m . c^m \dots = (a . b . c \dots)^m$, since in both cases the same number of factors equal to a, equal to b, &c., are multiplied.

Fractional exponent.

But if the exponent becomes fractional, for instance, $\frac{m}{s}$; and, consequently, the preceding powers are changed into $a^{\frac{m}{s}}, b^{\frac{m}{s}}, c^{\frac{m}{s}}, \dots$. Let, as before, ν represent the numerical value, which multiplied s times by itself gives a, and let $\gamma, \delta, \dots$, be the numerical values or numbers, which multiplied, each s times by itself, give $b, c, \dots$, we will have

$$a^{\frac{m}{s}} = \nu^m,\ b^{\frac{m}{s}} = \gamma^m,\ c^{\frac{m}{s}} = \delta^m, \dots.$$

and consequently,

$$a^{\frac{m}{s}} . b^{\frac{m}{s}} . c^{\frac{m}{s}} \dots = \nu^m . \gamma^m . \delta^m \dots = (\nu . \gamma . \delta . .)^m.$$

Now, since s factors equal to $\nu . \gamma . \delta \dots$ give $a . b . c \dots$ the last number of this equation represents the power $(a . b . c \dots)^{\frac{m}{s}}$. Therefore,

$$a^{\frac{m}{s}} . b^{\frac{m}{s}} . c^{\frac{m}{s}} \dots = (a . b . c \dots)^{\frac{m}{s}}.$$

General inference and rule.

Hence, we generally infer, that

The product of any number of powers of the same degree, either whole or fractional, is equal to the product of the roots raised to the same power.

Powers of powers: case of the exponents whole numbers.

§ 44. Let now the power a^m be taken as the root of another power of the degree h. Supposing h a whole number, $(a^m)^h$ signifies that the product of m factors equal to a is multiplied h times by itself. But this comes to the same as to take the product of $m.h$ factors equal to a, which product is expressed by the power $a^{m.h}$; hence, $(a^m)^h = a^{m.h}$.

Fractional exponents.

The same multiplication of exponents will take place when they become fractional numbers. Suppose, in fact, m to be changed into $\frac{m}{s}$, and h into $\frac{h}{r}$, or the power $a^{\frac{m}{s}}$ raised to the power $\frac{h}{r}$. And making again, as in preceding numbers $a^{\frac{m}{s}} = \nu^m$, we will have

$$\left(a^{\frac{m}{s}}\right)^{\frac{h}{r}} = (\nu^m)^{\frac{h}{r}} \dots . (o).$$

Let now β be another number which multiplied r times by itself gives ν; in this supposition, we have

$$\nu = \beta^r;$$

and, consequently, $\nu^m = \beta^{r.m} = \beta^{m.r} = (\beta^m)^r$.
From this we infer that β^m is a number which multiplied r times by itself, gives ν^m.

Resume now, again, the second member of the equation (o). To obtain the power expressed by the monomial $(\nu^m)^{\frac{h}{r}}$, it is enough to raise to the power h, the number which multiplied r times by itself, gives ν^m, but this number is β^m, therefore,

$$(\nu^m)^{\frac{h}{r}} = (\beta^m)^h = \beta^{m.h};$$

but $\nu^m = a^{\frac{m}{s}}$; hence,

$$\left(a^{\frac{m}{s}}\right)^{\frac{h}{r}} = \beta^{m.h} \dots . . (o,)$$

Now, since β^r is equal to ν, we will have, also

$$\beta^{r.s} = \nu^s;$$

but ν is that number, which multiplied s times by itself, gives a therefore,

$$\beta^{r.s} = a;$$

that is, β is such a number, which multiplied $r.s$ times by itself, gives a. Hence, the power $a^{\frac{m.h}{s.r}}$ is obtained by raising β to the power $m.h$, that is,

$$\beta^{m.h} = a^{\frac{m.h}{s.r}} = a^{\frac{m}{s}.\frac{h}{r}}.$$

Substituting, finally, this value in the second number of (o,) we have

$$\left(a^{\frac{m}{s}}\right)^{\frac{h}{r}} = a^{\frac{m}{s}.\frac{h}{r}} = a^{\frac{mh}{sr}};$$

General inference and rule.

as for the case of exponents, whole numbers. Hence, generally,

The power of a power of any root is equal to the same root, having for exponent the product of the given separate exponents.

Simplification of exponential fractions or quotients.

§ 45. We have seen already (37) how a quotient or fraction may be reduced to a simpler form by the elimination of common factors. Let us see here how in the same fraction we may change, in all cases, either the numerator or denominator into unity.

First case. Exponents whole numbers, and higher degree of the numerator.

Let us commence with the most simple case of the whole numbers m and n, and the first greater than the second. The fraction $\frac{a^m}{a^n}$ may be easily transformed in this case to the form of a whole quantity, because, calling d, the difference $m-n$, we have $a^d = a^{(m-n)}$, but (42), $a^m = a^{(m-n)} . a^n$; hence, $a^m = a^d . a^n$,

$$\frac{a^m}{a^n} = \frac{a^d . a^n}{a^n} = a^d = a^{(m-n)}.$$

Second case. Higher degree in the denominator.

Suppose now, the given fraction to be $\frac{a^n}{a^m}$ we will have then

$$\frac{a^n}{a^m} = \frac{a^n}{a^d a^n} = \frac{1}{a^d} = \frac{1}{a^{m-n}}.$$

Third case. Fractional exponents, and the higher in the numerator.

Similar modifications will take place with fractional exponents. Suppose, in fact, $\frac{m}{s}$ and $\frac{n}{s}$ to be the exponents, then $\frac{m}{s} - \frac{n}{s} = \frac{d}{s}$, and $a^{\left(\frac{m}{s}-\frac{n}{s}\right)} . a^{\frac{n}{s}} = a^{\frac{m}{s}}$; therefore, $a^{\frac{m}{s}} = a^{\frac{d}{s}} . a^{\frac{n}{s}}$, and

$$\frac{a^{\frac{m}{s}}}{a^{\frac{n}{s}}} = \frac{a^{\frac{d}{s}} . a^{\frac{n}{s}}}{a^{\frac{n}{s}}} = a^{\frac{d}{s}} = a^{\left(\frac{m}{s}-\frac{n}{s}\right)}.$$

Fourth case. Higher degree in the denominator.

If the given fraction is $\frac{a^{\frac{n}{s}}}{a^{\frac{m}{s}}}$, then we have

$$\frac{a^{\frac{n}{s}}}{a^{\frac{m}{s}}} = \frac{a^{\frac{n}{s}}}{a^{\frac{d}{s}} . a^{\frac{n}{s}}} = \frac{1}{a^{\frac{d}{s}}} = \frac{1}{a^{\frac{m}{s}-\frac{n}{s}}}.$$

Fifth case. Equal exponents. Following the analogy of the preceding cases, the algebraic fraction $\frac{a^m}{a^m}$, will give either a^{m-m}, or $\frac{1}{a^{m-m}}$; but $\frac{a^m}{a^m} = 1$; therefore, $a^{m-m} = \frac{1}{a^{m-m}} = 1$, or

$$a^0 = 1.$$

Negative exponents. Extending still farther the analogy, and supposing in the quotients $\frac{a^n}{a^m}, \frac{a^m}{a^n}$ $m > n$, as above; as from the latter, we infer $\frac{a^m}{a^n} = a^{m-n} = a^d$, so we represent the former $\frac{a^n}{a^m}$ by $a^{n-m} = a^{-d}$. And likewise the former fraction gives $\frac{a^n}{a^m} = \frac{1}{a^{m-n}} = \frac{1}{a^d}$, so by analogy, we write $\frac{a^m}{a^n} = \frac{1}{a^{-d}}$. Therefore the quotient $\frac{a^m}{a^n}$ is represented at once by a^d, and by $\frac{1}{a^{-d}}$, and the quotient $\frac{a^n}{a^m}$ by $\frac{1}{a^d}$, and by a^{-d}, or the expressions

$$a^d, \frac{1}{a^{-d}},$$

$$a^{-d}, \frac{1}{a^d},$$

are considered as synonyms.

It is not necessary to extend here the same observations to the case of fractional exponents, to which they could be as easily applied, as it is evident.

Inferences. From the said convention it follows, first, that

$$a^m . a^{-m} = 1.$$

Secondly, since the difference d between any two numbers m, n, has the same absolute value, but different signs, according as m is subtracted from n, or *vice versa*, it follows, that in all cases we may write:

$$\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}}.$$

Examples.

§ 46. Write in the upper or lower line all the terms of the given fractions.

(1.) $\frac{a^2b}{c^2d^3}$, Answers: $a^2bc^{-2}d^{-3}$, or $\frac{1}{a^{-2}b^{-1}c^2d^3}$.

(2.) $\frac{a^2b^{-2}}{f^2d^3}$, $a^2b^{-2}f^{-2}d^{-3}$, or $\frac{1}{a^{-2}b^2f^2d^3}$.

(3.) $\frac{a^mc^n}{d^pe^q}$, $a^mc^nd^{-p}e^{-q}$, or $\frac{1}{a^{-m}c^{-n}d^pe^q}$.

(4.) $\frac{a^{m-n}\,.\,d^p}{g^mf}$, $a^{m-n}.\,d^p.\,g^{-m}f^1$, or $\frac{1}{a^{n-m}d^{-p}g^mf}$.

(5.) $\frac{d^{-m}e^{-n}}{a^2b^3}$, $d^{-m}e^{-n}a^{-2}b^{-3}$, or $\frac{1}{d^me^na^2b^3}$.

Write the following expressions all with positive exponents, and reduce the exponential quantities according to the preceding rules:

	Given.	Answers.
(6.)	$\frac{a^{(m-s)}\,.\,b^{(n-r)}\,.\,c^2}{a^{-s}b^{-r}c^{-3}}$,	$a^m\,.\,b^n\,.\,c^5$.
(7.)	$\frac{a^{-q}b^{-r}}{a^mb^n}$,	$\frac{1}{a^{m+q}\,.\,b^{n+r}}$.
(8.)	$\frac{a^md^{-n}}{a^{-n}b^{-m}}$,	$\frac{a^{m+n}\,.\,b^m}{d^n}$.
(9.)	$\frac{d^{-m}\,f^n}{f^{-q}d^r}$,	$\frac{f^{(q+n)}}{d^{(m+r)}}$.
(10.)	$\frac{a^mb^{-n}d^3}{f^ng^{-m}}$,	$\frac{(a\,.\,g)^md^3}{(f\,.\,b)^n}$.
(11.)	$\frac{b^{(m-n)}f^{-n}}{b^{-n}l^{-m}}$,	$\frac{(b\,.\,l)^m}{f^n}$.

	Given.	Answers.
(12.)	$\dfrac{a^n . b^{-(m+n)} . c^{(m-2n)} d^{-m}}{a^{(n-m)} . b^{-(n+2m)} . c^{-(n-m)} d^{-(m-n)}}$,	$\dfrac{a^m . b^m}{c^n . d^n}$.
(13.)	$\dfrac{a^{-4} b . d^n . f^{-r} . p^{-(r-m)}}{a^{-6} . b^{-1} . d^{-(m-n)} q^r . p^m}$,	$\dfrac{(a . b)^2 . d^m}{(f . p . q)^r}$.
(14.)	$\dfrac{a^s p^{-s} (f^t)^m . (f^t)^{-r}}{b^{-s} . p^t . (f^r)^{-t}}$,	$\dfrac{(a . b)^s f^{mt}}{p^{s+t}}$.
(15.)	$\dfrac{a^m . b^{-(n-m)}}{f^{-s} . a^{-m} b^{-(n+m)}}$,	$(a . b)^{2m} . f^s$.
(16.)	$\dfrac{a^m . b^{(s-q)} f^{-r}}{a^{-2m} b^{-q} f^r}$,	$\dfrac{(a^m)^3 b^s}{(f^r)^2}$.

There is also another modification of algebraic expressions of this kind. Observe that the power of any fractional monomial, for instance, $\left(\frac{a}{b}\right)^m$ is equal to the power m of the numerator, divided by the power m of the denominator; since $\frac{a}{b} \cdot \frac{a}{b} \ldots$ is equal to the product of a taken m times, divided by the product of b taken also m times. The expression then $\left(\frac{a}{b}\right)^m$ may be transformed into $\frac{a^m}{b^m}$, or *vice versa*. So, for example, we may write

$$\frac{a^m . b . q}{b^{m+1}} = \left(\frac{a}{b}\right)^m . q,$$

$$\left(\frac{a}{b}\right)^n \cdot \frac{b(q . r)^2}{f . q^{(n+4)}} = \frac{a^n}{b^{(n-1)} f q^n} \times \left(\frac{r}{q}\right)^2, \text{ \&c.}$$

The extraction of roots is the inverse operation of raising to power.

§ 47. EXTRACTION OF ROOTS.—To extract the root of a quantity a is to find out another quantity r, which raised to a certain power, gives a. Hence, the given quantity a is considered as the power of another quantity r, raised to the exponent indicated by the root; and since the extraction of the root consists in finding this r, the operation, therefore, is the inverse of raising to power.

Conventional signs and nomenclature.

The manner in which this operation is indicated is by the expression

$$\sqrt[m]{a},$$

which is read m^{th} root of a, and with generical terms *radical expression*, or simply *radical*. So, likewise, $\sqrt{\ }$ is called *radical sign*, m *index* or *degree* of the root, which is the exponent to be given to the unknown quantity r or root, to obtain a. If m is equal to 2, the root is called *square*, and in this case the index is omitted, so that the expression $\sqrt{a}$ without any number in the radical sign, signifies square root of a. When m is equal to 3, the root is termed *cubical*, $\sqrt{m}$, is also represented by $a^{\frac{1}{m}}$, and $\sqrt[n]{a^m}$ by $a^{\frac{m}{n}}$,* fractional powers on which we may operate, as on whole powers.†

Square and cubical roots.

The root of a quantity may be always represented by a fractional exponent given to the same quantity.

§ 48. Suppose m n to be whole numbers, we say that $\sqrt[n]{a^m}$ the n^{th} root of a^m is equal to $a^{\frac{m}{n}}$. Because, let α be the numerical value which multiplied n times by itself, gives a, we have

$$\alpha^m = a^{\frac{m}{n}};$$

and, consequently, $(\alpha^m)^n = \left(a^{\frac{m}{n}}\right)^n = a^m.$

Therefore, α^m is that numerical value, which raised to the power n, gives a^m; hence, α^m is the n^{th} root of a^m, but $\alpha^m = a^{\frac{m}{n}}$, therefore,

$$\sqrt[n]{a^m} = a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}};$$

and supposing $m = 1$,

$$\sqrt[n]{a} = a^{\frac{1}{n}};$$

that is, the n^{th} root of a, is a raised to the power $\frac{1}{n}$, and *vice versa*, (40.)

Case of the fractional index.

Observe, also, that

$$(\alpha^m)^{\frac{n}{m}} = \alpha^n;$$

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* See the following number.

† §§ 42, 43, 44, 45.

A radical having a fractional index can be transformed into one having a whole number for index.

consequently,

$$\sqrt[\frac{n}{m}]{a^n} = \sqrt[\frac{n}{m}]{(a^m)^{\frac{n}{m}}} = a^m;$$

but $a^n = a$ and $a^m = a^{\frac{m}{n}}$; again, $a^{\frac{m}{n}} = \sqrt[n]{a^m}$, so from the last equation, we infer

$$\sqrt[\frac{n}{m}]{a} = a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

That is, the $\left(\frac{n}{m}\right)^{\text{th}}$ root of a is equal to the $\left(\frac{m}{n}\right)^{\text{th}}$ power of the same a, and equal to the n^{th} root of a^m.

Products of roots.

§ 49. Hence, (42, 43,)

$$\sqrt[\frac{s}{m}]{a} \,.\, \sqrt[\frac{s}{t}]{a} \,.\, \sqrt[\frac{s}{r}]{a} \ldots = \sqrt[\frac{s}{m+t+r+\cdots}]{a}.$$

$$= \sqrt[s]{a^{m+t+r+\cdots}}.$$

First case. Different roots of the same quantity.

Because,

$$\sqrt[\frac{s}{m}]{a} \,.\, \sqrt[\frac{s}{t}]{a} \,.\, \sqrt[\frac{s}{r}]{a} \ldots = a^{\frac{m}{s}} \,.\, a^{\frac{t}{s}} \,.\, a^{\frac{r}{s}} \ldots$$

$$= a^{\frac{m}{s}+\frac{t}{s}+\frac{r}{s}+\cdots}$$

$$= a^{\frac{m+t+r+\cdots}{s}},$$

$$= \&c.$$

And, since fractions may be always reduced to the same numerator, no less than to the same denominator, similar reductions can be performed in all cases; and, supposing in the preceding formulas $s = 2$, we will have

$$\sqrt[\frac{2}{m}]{a} \,.\, \sqrt[\frac{2}{t}]{a} \,.\, \sqrt[\frac{2}{t}]{a} \ldots = \sqrt{a^{m+t+r+\cdots}}.$$

Second case. Equal roots of different quantities.

In the case of equal roots of different quantities, we will have

$$\sqrt[\frac{s}{m}]{a} \,.\, \sqrt[\frac{s}{m}]{b} \,.\, \sqrt[\frac{s}{m}]{c} \ldots = \sqrt[\frac{s}{m}]{a \,.\, b \,.\, c \ldots}$$

Because,

$$= \sqrt[s]{(a \,.\, b \,.\, c \ldots)^m},$$

$$\sqrt[\frac{s}{m}]{a} \,.\, \sqrt[\frac{s}{m}]{b} \,.\, \sqrt[\frac{s}{m}]{c} \ldots = a^{\frac{m}{s}} \,.\, b^{\frac{m}{s}} \,.\, c^{\frac{m}{s}} \ldots$$

$$= (a \,.\, b \,.\, c \ldots)^{\frac{m}{s}};$$

and, supposing $m = 1$.

$$\sqrt[s]{a}\,.\,\sqrt[s]{b}\,.\,\sqrt[s]{c}\ldots = \sqrt[s]{a\,.\,b\,.\,c\ldots}$$

Root of a root.

Let now the root of a root be given, for instance, $\sqrt[\frac{s}{m}]{\sqrt[\frac{r}{h}]{a}}$, we will find

$$\sqrt[\frac{s}{m}]{\sqrt[\frac{r}{h}]{a}} = \sqrt[\frac{sr}{mh}]{a};$$

because, (44) $\sqrt[\frac{s}{m}]{\sqrt[\frac{r}{h}]{a}} = \sqrt[\frac{s}{m}]{a^{\frac{h}{r}}} = \left(a^{\frac{h}{r}}\right)^{\frac{m}{s}} = a^{\frac{hm}{rs}}.$

Supposing the denominators m and h equal to unity, then

$$\sqrt[s]{\sqrt[r]{a}} = \sqrt[sr]{a}.$$

The index of the root, and that of the quantity under the radical sign, may be multiplied or divided by the same quantity.

Observe here also, that, since

$$\sqrt[n]{a^m} = a^{\frac{m}{n}} \text{ and } a^{\frac{m}{n}} = a^{\frac{m.k}{n.k}} = \sqrt[nk]{a^{mk}},$$

so we have also

$$\sqrt[n]{a^m} = \sqrt[nk]{a^{m.k}};$$

again, $a^{\frac{m}{n}} = \left(a^{\frac{m}{k}}\right)^{\frac{k}{n}} = \sqrt[\frac{n}{k}]{a^{\frac{m}{k}}}$; hence,

$$\sqrt[n]{a^m} = \sqrt[\frac{n}{k}]{a^{\frac{m}{k}}}.$$

Four cases embraced by a radical expression.

§ 50. Let us now analyze the formula,

$$\sqrt[n]{a} = \alpha.$$

Four cases may occur about this formula. First, n (which is supposed to be a whole number) is either an even number $= 2i$, or an odd number $= 2i + 1$: again, with $n = 2i$, we can suppose a positive or a negative, and the same supposition can be made when $n = 2i + 1$.

First case.

Let us commence to examine the case of a positive, and $n = 2i$. Since the formula $\sqrt[n]{a} = \alpha$, supposes $\alpha^n = a$, we will have

$$\alpha^{2i} = a.$$

But $+\alpha$, as well as $-\alpha$, raised at the power $2i$, give the same positive product; hence,

$$(+\alpha)^{2i} = (-\alpha)^{2i} = a.$$

Therefore, in the first case, we have

$$\sqrt[2i]{+a} = \pm \alpha;$$

that is, the $(2i)^{\text{th}}$ root of a is either positive or negative.

Second case. Let again a be positive, but the index $n = 2i + 1$, or let $\sqrt[2i+1]{+a} = \alpha$, we will have $(\alpha)^{2i+1} = +a$, but α in this case must be necessarily positive, because, $2i + 1$, factors of the same quantity, cannot give a positive product unless the same quantity is positive; hence,

$$(+\alpha)^{2i+1} = a,$$

and

$$\sqrt[2i+1]{+a} = +\alpha.$$

Third case. Let now a be negative, and the index n still equal to $2i + 1$. Since a negative product of the same quantity multiplied $2i + 1$ times, can be obtained only by a negative quantity, so we will have, in this case

$$(-\alpha)^{2i+1} = -a,$$

and

$$\sqrt[2i+1]{-a} = -\alpha.$$

Fourth case. The last case is that of a negative when $n = 2i$ and expressed by $\sqrt[2i]{-a}$. But neither among positive nor negative numbers or quantities, any one is to be found representing this root, because we have seen in the first case, that with a either positive or negative, we have always $\alpha^{2i} = +a$; hence the radical expression

$$\sqrt[2i]{-a},$$

Imaginary radicals or roots. is termed *imaginary* root, or radical or imaginary quantity or expression.

Although such expressions, considered in one point of view, are paradoxical, yet they may be also considered as symbols of terms heterogeneous to real quantities, that is, $\sqrt{-a}$ is the symbol of a term not included in the category of real quantities, which term being incapable of taking a real form, cannot be represented but by an ambiguous or paradoxical form. These mysterious symbols, so frequently met with in mathematical investigations, are used and profitably managed like real quantities.

Operations on imaginary quantities. § 51. Let us then see here some of the elementary operations about such quantities, and first, let us take

Multiplication and division. the product of $\sqrt{-a}$ by $\sqrt{-b}$. Since $-a$ and $-b$ are equivalent to $a \times -1$, $b \times -1$, and $\sqrt{-a}$, $\sqrt{-b}$ are equivalent to $-a^{\frac{1}{2}}$, $-b^{\frac{1}{2}}$, so we will have

$$\sqrt{-a} = (a \times -1)^{\frac{1}{2}} = a^{\frac{1}{2}}.-1^{\frac{1}{2}} = \sqrt{a}.\sqrt{-1},$$

$$\sqrt{-b} = (b \times -1)^{\frac{1}{2}} = b^{\frac{1}{2}}.-1^{\frac{1}{2}} = \sqrt{b}\ \sqrt{-1}.$$

Therefore, $\sqrt{-a}\sqrt{-b} = \sqrt{a}\sqrt{b}(\sqrt{-1})^2$;

but $\sqrt{a}.\sqrt{b} = \sqrt{ab}$, $(\sqrt{-1})^2 = -1$; hence

$$\sqrt{-a}.\sqrt{-b} = -\sqrt{ab}.$$

In like manner, we will have

$$-a\sqrt{-1}\ .\ b\sqrt{-1} = a.b;$$

because $-a.b = -ab$ and $(\sqrt{-1})^2 = -1$.

Again, the ratio $a\sqrt{-1} : b\sqrt{-1}$, gives

$$\frac{a\sqrt{-1}}{b\sqrt{-1}} = \frac{a}{b}.$$

Powers. Since $(\sqrt{-1})^2 = -1$,
we will have, also

$$(\sqrt{-1})^4 = (-1)^2 = +1,$$

$$(\sqrt{-1})^6 = (-1)^3 = -1,$$

and so on. And generally,

$$(\sqrt{-1})^{2i} = (\sqrt{(-1)}^2)^i = (-1)^i = \pm 1,$$

in which the sign is positive when i is an even number. In an equal manner, we will have

$$(\sqrt{-1})^{2i+1} = \pm\sqrt{-1},$$

where, likewise, the positive sign is to be taken with i an even number. Because, $(\sqrt{-1})^{(2i+1)} = (\sqrt{-1})^{2i}.\sqrt{-1}$; hence,

$$(\sqrt{-1})^{(2i+1)} = \pm 1\sqrt{-1} = \pm\sqrt{-1}.$$

A corollary follows from this doctrine, applicable to the case of real quantities. We have seen (50) that the n^{th} real root of a is either double or only one or none; but if together with the real, we reckon also the imaginary roots of the same quantity, their number is quite different. So, for example,

$$(+2)^4, (-2)^4, (+2\sqrt{-1})^4, (-2\sqrt{-1})^4,$$

give all $+16$; hence, the fourth root of 16 admits four different values, two real, and two imaginary—namely,

$$\sqrt[4]{16} = \pm 2,$$
$$\sqrt[4]{16} = \pm 2\sqrt{-1}.$$

Irrational radicals and quantities.

§ 52. It now remains for us to add some remarks concerning irrational radicals. These remarks are connected with other questions somewhat foreign to the subject of the preceding numbers. For this reason they have been left for the last of the present article. We will commence by illustrating the method of finding the greatest common measure of two numbers.

Greatest common measure.

Suppose M and N to be two whole numbers, or numerical values of two quantities, and $M > N$. Let the whole number q' be the quotient of M divided by N and R' the remainder; or, in other terms, let M*b* be equal to q' times N plus R', which is less than N. Dividing now N by R', let q'' be the quotient and R'' the remainder; dividing in the next place, R' by R''', let q''' be the quotient, and R''' the remainder, &c., in this manner, we have the equations

$$\left.\begin{array}{l} M = q'\,N + R' \\ N = q''\,R' + R'' \\ R' = q'''R'' + R''' \\ R'' = q^{iv}R''' + R^{iv} \\ \qquad \&c. \end{array}\right\} (o._1)$$

Now, if R^{v}, for example, would be found equal to zero, namely, if R^{iv} divides exactly R''' without any remainder, we say first, that the same R^{iv} is an exact divisor of M and N; and secondly, this divisor is the greatest common divisor or measure of M and N. The first assertion is easily demonstrated, observing that R^{iv} cannot exactly divide R''', without dividing also R'', because, $R'' = q^{iv}R''' + R^{iv}$; but by supposition, $R''' = q^{v}R^{iv}$; therefore, $R'' = q^{iv}.\,q^{v}.\,R^{iv} + R^{iv}$; that is to say, R'' is equal to $q^{iv}.\,q^{v} + 1$ times R^{iv}, and for brevity's sake, calling Q the sum, $q^{iv}.\,q^{v} + 1$: $R'' = QR^{iv}$, or $\frac{R''}{R^{iv}} = Q$.

But, again, R^{iv} cannot be an exact divisor of R'' without dividing also R'. This remainder is equal to $q'''R'' + R'''$, but by supposition, $R''' = q^{v}R^{iv}$, and consequently, $R'' = QR^{iv}$; hence, $R' = q'''\,QR^{iv} + q^{v}R^{iv}$; that is to say, R' is equal to $q'''.Q + q^{v}$ times R^{iv}, and for brevity's sake, calling Q' the sum $q'''Q + q^{v}$, $R' = Q'R^{iv}$, or $\frac{R'}{R^{iv}} = Q'$. Following the same process, we find that N and M are, likewise, multiples of the same number R^{iv}, which, consequently, is an exact divisor of them. But the same remainder is the greatest common divisor of the same number; because, no number—K for instance—greater than the last said remainder, can divide both numbers M and N.

To demonstrate this, observe, first, that subtracting from both members of the preceding equations, the same number, namely, $q'N$, from the members of the first, $q''R'$; from those of the second, and so on, we easily deduce (16,) the following equations:

$$\left.\begin{aligned} R' &= M - q'N \\ R'' &= N - q''R' \\ R''' &= R' - q'''R'' \\ R^{iv} &= R'' - q^{iv}R''' \\ &\ \&c. \end{aligned}\right\} (o_1.)$$

Making now the supposition that K divides exactly M and N, we must admit that it divides also R'. Because M, for example, will contain exactly K two or three, or r times, and N will exactly contain the same K, for instance, s times; then $M = rK$, $N = sK$, and $M - q'N = rK - q'sK$, but from the last equations, $M - q'N = R'$; hence, $R' = rK - q'sK$; that is to say, R' contains $(r - q's)$ times K, or R' is an exact multiple of R'. Reasoning in the same manner, since N and R' are multiples of K, it follows from the second equation, that K must be an exact divisor of R'' also, and for the same reason an exact divisor of R''' and R^{iv}; but K is greater than

R^{iv}, and consequently, R^{iv} cannot be divided by K; hence, the supposition of K a common measure of M and N, and greater than the last remainder cannot be admitted; and we generally infer, that when two numbers M and N are given, the last remainder found in the above-described process is their greatest common measure.

Examples. Let $M = 189$, $N = 147$, we will have

$$\frac{M}{N} = \frac{189}{147} = 1 + \frac{42}{147} \qquad \text{Namely, } q' = 1,\ R' = 42,$$

$$\frac{N}{R'} = \frac{147}{42} = 3 + \frac{21}{42}. \qquad \text{“} \qquad q'' = 3,\ R'' = 21,$$

$$\frac{R'}{R''} = \frac{42}{21} = 2 + 0. \qquad \text{“} \qquad q''' = 2,\ R''' = 0.$$

The last remainder, therefore, is $R'' = 21$, the greatest common measure of the numbers 189, 147.

Let $M = 154$, $N = 15$, we will have

$$\begin{array}{lr|l}
M & = \ldots\ldots 154 & \\
N & = \ldots\ldots\ 15 & 10\ldots\ldots = q' \\
R' & = \ldots\ldots\ \ 4 & \ 3\ldots\ldots = q'' \\
R'' & = \ldots\ldots\ \ 3 & \ 1\ldots\ldots = q''' \\
R''' & = \ldots\ldots\ \ 1 & \ 3\ldots\ldots = q^{iv} \\
R^{iv} & = \ldots\ldots\ \ 0 &
\end{array}$$

In this case unity is the last remainder; hence, no number above unity, divides exactly both 154 and 15.

Prime numbers to each other. When numbers have no common measure, except unity, they are called *prime to each other*. But the numbers of the present example are each divisible by other numbers above unity, the first by 2, 7, and 11; the second, by 3 and 5. But there are numbers which, even separately considered, cannot be exactly divided except by unity, such as 3, 5, 7, 13, 19, 31, &c.

Prime numbers in themselves. These numbers are termed *prime in themselves*, or simply *prime numbers*.

Having premised these principles, let us pass to demonstrate the following proposition :

How a prime number can accurately divide the product of two other numbers.

§ 53. *A prime number* N *which accurately divides the product* MP, *must necessarily divide one of the factors* P, M. Suppose, first, $N < M$, and M not accurately divisible by N. With regard to these two numbers we will have the same equations, (o), (o_1) of the preceding number, and, besides, the last remainder must be equal to 1. Multiplying now both members of the equations (o_1) by P, we have

$$\left.\begin{array}{l} PR' = MP - q'NP \\ PR'' = NP - q''R'P \\ PR''' = R'P - q'''R''P \\ PR^{iv} = R''P - q^{iv}R'''P \\ \quad\&c. \end{array}\right\} (o_2.)$$

Now N, by supposition, divides accurately MP; and since $q'\frac{NP}{N} = q'P$, it divides also $q'NP$; therefore, the second number of the first equation (o_2) is accurately divisible by N, and consequently also the first PR′. In the second member of the next equation, we find NP again, and PR′, both exactly divisible by N; hence, the first member PR″ of this equation also is exactly divisible by N. In equal manner we prove that the following products PR''', PR^{iv}, ... are all exactly divisible by N. But in our present supposition the last remainder must be equal to 1; and consequently the last product exactly divisible by N is $P.1$, or P.

Suppose now $N > M$, and divide N by M, and again, M by the first remainder, and this by the second, and so on; call g', g'', g''', ... the quotients, and ρ', ρ'', ρ''', ... the remainder, the last of which must be as in the preceding case, equal to 1, on account of N being a prime number. With these elements we can easily form equations similar to the preceding, and those corresponding to the equations (o_2), will be

$$\left.\begin{aligned} P\rho' &= NP - q'MP \\ P\rho'' &= MP - q''\rho'P \\ P\rho''' &= \rho'P - q'''\rho''P \\ &\&c. \end{aligned}\right\} (o_3.)$$

Now N, which certainly divides exactly NP, divides by supposition, accurately, MP; therefore, the second member of the first (o_3) is exactly divisible by N; hence, also, its equivalent $P\rho'$. Reasoning as in the preceding case, we will find all the following products, $P\rho''$, $P\rho'''$, ... exactly divisible by N; but the last ρ is equal to 1, and consequently the last of those products exactly divisible by N is P. This factor, therefore, is exactly divisible by N, when the other factor M, either greater or less than N, is not divisible by the same prime numbers. It is plain that in the same manner in which we have demonstrated that P is exactly divisible by N, when the other factor M is not divisible by the same N; so we could demonstrate that when P is not divisible by N, then M is certainly divisible by it; and we can generally infer, therefore, that when the product PM is accurately divisible by N, a prime number, and one of the factors is not divisible by it, the other is necessarily divisible by the same number N. We may observe, that even in the case in which N is not a prime number absolutely, but prime only to M; and N divides exactly the product MP, P is exactly divisible by N. Which thing is proved in the same manner as the preceding theorem.

A prime number dividing a product, divides at least one of the factors.

Corollary.—We can now easily infer this corollary: When a prime number N divides exactly the product $Q = R.M.K.H..$, the same must necessarily divide at least one of the factors.

Call P' the partial product $M.K.H...$, and P'' the partial product $K.H...$; P''' the product $H....$, and so on, we will have the given

$$\begin{aligned} Q &= R.P' \\ P' &= MP'' \\ P'' &= KP''' \\ P''' &= HP^{\text{iv}}, \text{ \&c.} \end{aligned}$$

Corollaries.

Now R, and the first factor of each one of the partial products, is one of the factors of our product $Q = R.M.K.H \ldots$, the number of which is filled by the last of those partial products which contains the two last factors of Q. Suppose now that none of the first factors R, M, K, ... is exactly divisible by N; it follows first, that P′ is divisible by N, and consequently the second product MP″; but M by supposition is not divisible by N, therefore the factor P″ must be a multiple of N; and consequently the product KP‴, and the factor P‴ of this product, and so on, till the last factor of the last product, which is one of the factors of Q. If none, therefore, of the factors R.M.K .... of the given product Q, until the last, is divisible by N, the last must certainly be divisible by it. *Vice versa*, if none of the factors of Q is divisible by the prime number N, neither Q can be divisible by it. It follows, besides, that if M or K are not divisible by a prime number N, neither M^n and K^n are divisible by it; because n representing a whole number, M^n and K^n are the products of n factors, none of which is exactly divisible by N.

Powers of fractional expressions.

Irreducible to a simple form.

§ 54. Let us now take the fractional expression $\frac{a}{b}$ reduced to its simplest terms: that is, to such simple elements a and b, as to admit no common divisor except unity. We say that the numerical values of the powers $\frac{a^2}{b^2}, \frac{a^3}{b^3}, \ldots \frac{a^n}{b^n}$ are necessarily fractional numbers, irreducible to a simpler form: that is to say, admitting no common measure except unity. Suppose, in fact, that a^n and b^n are exactly divisible by a number greater than 1; in this case a also and b must have a common measure above unity, which is against the present supposition. Before we demonstrate this, let us observe first, that whole numbers are either prime numbers or products of prime numbers; consequently we cannot suppose a number divisible by another, which is not prime, without supposing that same number divisible also by some prime number. Because, let the whole number N be divisible by the other whole number $P = m.n$: that is, let the quotient $\frac{N}{P} = Q$ be a whole number; since from this equation we deduce the other $Q.P = N$, or $Q.m.n = N$, we will have also, $\frac{N}{m} = Q.m$, $\frac{N}{n} = Q.m$; that is, N is exactly divisible by the factors of P.

Let us now resume our fraction $\frac{a^n}{b^n}$, the supposed common measure is either a prime number or not; in both cases we must admit that some prime number is common divisor to both a^n and b^n; but we have already seen that a^n or b^n cannot be divisible by any prime number, unless a and b are divisible by the same number. Therefore, when the fractional expression $\frac{a}{b}$ is reduced to its simplest terms, any power $\frac{a^n}{b^n}$ of the same fraction is another fractional expression, whose terms cannot be reduced to a simpler form, and consequently it is essentially fractional.

Irrational roots.

§ 55. We are now able to see how, among the radical quantities, there are some whose numerical values can never be exactly assigned, and are, consequently, to be reckoned among the irrational expressions.

Powers of the natural numbers.

Observe, that taking the squares, the cubes, the fourth powers, and so on, of the natural series 1, 2, 3, 4, 5 of numbers, we have

Squares......... 1, 4, 9, 16, ...
Cubes.......... 1, 8, 27, 64, ...
Fourth powers. 1, 16, 71, 256, ..., &c.

That is, the square root of 1 is 1, and the square root of 4 is 2; hence, the square roots of the numbers 2, 3 are between 1 and 2, or they are numbers of an essentially fractional form, however reduced to their lowest terms; but we have seen that a fractional expression raised to any power gives constantly a fractional result; no number, therefore, between 1 and 2, if squared, can give for power either the whole numbers 2 or 3; therefore the numbers 2 and 3 have neither their square roots among whole numbers nor among numbers of fractional form; these roots, therefore, cannot be exactly expressed by any number either whole or fractional, although we may represent them by numbers more and more approaching to their value; the same roots, therefore, are irrational quantities. The same is to be said of the square roots of the numbers between 4 and 9, between 9 and 16, &c.; the same of the cubical roots of the numbers between 1 and 8, between 8 and 27, &c.; the same of the fourth roots of the numbers between 1 and 16, between 16 and 71, and so on. And generally, we infer, that *when whole numbers have not their roots among other whole numbers, neither can they have them among fractional ones.*

A series of rational or assignable numbers may be conceived approaching constantly to any irrational root.

§ 56. Let us see now how an indefinite series of rational numbers can be conceived constantly approaching to any irrational root.

Divide unity into any number of small equal parts, which we will call ω. It is plain, first, that ω is a fraction greater or smaller, according to the less or greater number of parts into which unity is divided; secondly, not only ω, but 2ω, 3ω, 4ω, ... are all fractions until we take the whole number of them; thirdly, the difference between two successive terms of the series ω, 2ω, 3ω, ... is smaller in proportion to the greater number of parts into which unity has been divided; fourthly, representing by n any number the difference between $n + \omega$ and $n + 2\omega$, is the same as that between the simple terms ω and 2ω. Finally, if n represents a whole number, $n + \omega$, $n + 2\omega$, $n + 3\omega$, and so on, are all fractional numbers comprised between n and $n + 1$, until we add to n all the parts ω into which unity has been divided. Each one, therefore, of the numbers $n + \omega$, $n + 2\omega$, $n + 3\omega$, reduced to its lowest terms, must have the fractional form $\frac{a}{b}$; none of them, therefore, raised to any power, gives an exact whole number. Now the square of 2, the first term of the series $2, 2 + \omega, 2 + 2\omega, 2 + 3\omega, \ldots\ldots 2 + 1$ $(q,)$ is 4, and the square of the last $2 + 1$, is 9. The squares, therefore, of the included terms must be included between 4 and 9, and the square of the second term $2 + \omega$ is greater than 4, but nearer to it than the square of the next $2 + 2\omega$; the square of $2 + 3\omega$ is still greater, and so on. But increasing indefinitely the number of the particles ω, the difference of the squares of the successive terms diminishes also indefinitely; and in the same manner as by increasing the number of these particles, the square of the second term $2 + \omega$ approaches more and more to 4, so the squares of some of the following terms will more and more approach to 5, to 6, to 7, to 8, and the square of the term before the last will approach more and more to 9. But, in the same manner as this square, and that of the second term $2 + \omega$, never reach 9 and 4, however great may be the number of the particles ω, so none of the squares of the intermediate terms will ever reach the whole numbers 5, 6, 7, 8. Therefore, the radical expressions

$$\sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8},$$

and the same we say of all similar roots, are numbers which cannot be exactly expressed, and, consequently, neither measured; for this reason they are called *incommensurable* or *irrational*, and *surd*.

Beside these irrational numbers, other irrational quantities occur in mathematical questions, and all are reckoned among *transcendental* expressions.

Miscellaneous examples.

§ 57. Give the exponential form to the following roots: (See 47, 48.)

$$\sqrt{a}, \sqrt{a^m}, \sqrt[r]{a}, \sqrt[n]{a^r}, \sqrt[n]{a^{-m}}, \sqrt[m]{\frac{a}{b}}, \sqrt{\frac{1}{a^{-s}}}.$$

Exponential form given to roots.

Ans.

$$\sqrt{a} = a^{\frac{1}{2}}, \sqrt{a^m} = a^{\frac{m}{2}}, \sqrt[n]{a} = a^{\frac{1}{n}},$$

$$\sqrt[n]{a^r} = a^{\frac{r}{n}},\ \sqrt[n]{a^{-m}} = \frac{1}{a^{\frac{m}{n}}},\ \sqrt[m]{\frac{a}{b}} = \frac{a^{\frac{1}{m}}}{a^{\frac{1}{m}}},\ \sqrt{\frac{1}{a^{-2}}} = a.$$

Radical form given to powers.

Give the radical form to the following powers:

$$a^{\frac{n}{c}},\ \frac{a^{\frac{m}{n}}}{b^{\frac{m}{n}}},\ a^m,\ b^m a^{-m},\ \frac{a^{\frac{1}{2}}}{b^2}.$$

Observe, that $a^m = a^{\frac{nm}{n}}$, &c.

Ans. $a^{\frac{n}{c}} = \sqrt[c]{a^n},\ \frac{a^{\frac{m}{n}}}{b^{\frac{m}{n}}} = \sqrt[n]{\left(\frac{a}{b}\right)^m},\ a^m = \sqrt[n]{a^{nm}}$

$$b^m a^{-m} = \sqrt[n]{\left(\frac{b}{a}\right)^{nm}},\ \frac{a^{\frac{1}{2}}}{b^2} = \sqrt{\frac{a}{b^4}}.$$

Radicals reduced to the same degree.

Reduce the following radicals to the same degree. (See § 49.)

$$\sqrt[10]{\frac{a}{b}},\ \sqrt[5]{\frac{a^2}{b}},\ \sqrt{a^m b^{\frac{1}{10}}},$$

Ans. $\sqrt[10]{\frac{a}{b}},\ \sqrt[10]{\frac{a^4}{b^2}},\ \sqrt[10]{a^{10m} b}.$

$$\sqrt[8]{mn},\ \sqrt[3]{\frac{a}{b}},\ \sqrt[12]{m^2},\ \sqrt[6]{fd}.$$

Ans. $\sqrt[24]{(mn)^3},\ \sqrt[24]{\frac{a^8}{b^8}},\ \sqrt[24]{m^4},\ \sqrt[24]{(fd)^4}.$

$$\sqrt[m]{a^p},\ \sqrt[2]{a^q},\ \sqrt[n]{ab};$$

Ans. $\sqrt[2mn]{a^{2pn}},\ \sqrt[2mn]{a^{qmn}},\ \sqrt[2mn]{(ab)^{2m}}.$

Multiplication of radicals.

Examples of Multiplication. (See §§ 42, 43, 44, 45, 49.)

Given factors:

(1.) $\sqrt{\frac{a^{-n}}{c^n}},\ \sqrt{b^4 c^{-2}},\ \sqrt{b^{-3} c^{2n}},$

Given factors:

(2.) $\sqrt[n]{\frac{b^2}{f^{-1}.r^2}}, \sqrt[n]{\frac{r^{-2}b^{-5}}{a^nf^{-n}}}, \sqrt[n]{\frac{a^{2n}b^2}{f^2r^{-n}}}, \sqrt[n]{\frac{fr^4b}{b^{-n}}}.$

(3.) $\sqrt{\frac{a^{(2-n)}}{g}}, \sqrt{\frac{b^s}{g^{-1}b^{-4}}}, \sqrt{\frac{a^nb^{-s}}{b^6}}.$

(4.) $\sqrt[m]{a^4}, \sqrt[n]{\frac{c}{b}}.$

(5.) $\sqrt[3]{a^2}, \sqrt{a}, \sqrt[12]{a^{28}}.$

(6.) $\sqrt[nt]{\frac{a^{st}}{r^t.a^{-ln}}}, \sqrt[ng]{\frac{b^{fn}}{b^{-mg}}}, \sqrt[tg]{\frac{1}{a^{lg}.b^{ft}}}.$

Answers:

(1.) $\sqrt{\frac{a^{-a}}{c^n}} \times \sqrt{b^4c^{-2}} \times \sqrt{b^{-3}c^{2n}} = \sqrt{\frac{bc^n}{a^nc^2}}.$

(2.) $\sqrt[n]{\frac{b^2}{f^{-1}.r^2}} \times \sqrt[n]{\frac{r^{-2}.b^{-5}}{a^nf^{-n}}} \times \sqrt[n]{\frac{a^{2n}b^2}{f^2r^{-n}}} \times \sqrt[n]{\frac{fr^4b}{b^{-n}}}$
$= a.b.f.r.$

(3.) $\sqrt{\frac{a^{(2-n)}}{g}} \times \sqrt{\frac{b^s}{g^{-1}b^{-4}}} \times \sqrt{\frac{a^nb^{-s}}{b^6}} = \frac{a}{b}.$

(4.) $\sqrt[m]{a^4} \times \sqrt[n]{\frac{c}{b}} = \sqrt[mn]{\frac{a^{4n}c^m}{b^m}}.$

(5.) $\sqrt[3]{a^2} \times \sqrt{a} \times \sqrt[12]{a^{28}} = a^4.$

(6.) $\sqrt[nt]{\frac{a^{st}}{r^t.a^{-ln}}} \times \sqrt[ng]{\frac{b^{fn}}{b^{-mg}}} \times \sqrt[tg]{\frac{1}{a^{lg}.b^{ft}}} = \sqrt[n]{\frac{a^sb^m}{r}}.$

Reduction of the roots of roots.

Reduction of the roots of roots to a simpler form (§§ 44, 47.)

Given roots:

(1.) $\sqrt[3]{\sqrt[3]{a}}.$

(2.) $\sqrt{\sqrt[4]{b^{16}}}.$

(3.) $\sqrt[m]{\sqrt{a^{2m}}}$. (4.) $\sqrt[n]{\sqrt[n]{\frac{af^{n}}{b}}}$.

(5.) $\sqrt[4]{\sqrt[3]{\sqrt{\frac{a^{18}b^{16}}{l^7} \times \frac{f^{12}n^{-5}}{a^{-6}.b^{-8}}}}}$.

(6.) $\sqrt[m]{\sqrt[n]{\sqrt[p]{\left(\frac{a}{b}\right)^m \times \left(\frac{c}{d}\right)^n \times \left(\frac{e}{f}\right)^p \times \left(\frac{ac}{bd}\right)^p}}}$.

Answers.

(1.) $\sqrt[3]{\sqrt[3]{a}} = \sqrt[9]{a}$. (2.) $\sqrt{\sqrt[4]{b^{16}}} = b^2$.

(3.) $\sqrt[m]{\sqrt[n]{a^{2m}}} = \sqrt[n]{a^2}$. (4.) $\sqrt[n]{\sqrt[n]{\frac{af^{n}}{b}}} = \sqrt[n]{f} \cdot \sqrt[n^2]{\frac{a}{b}}$.

(5.) $\sqrt[4]{\sqrt[3]{\sqrt{\frac{a^{18}b^{16}}{l^7} \times \frac{f^{12}, ^{-5}}{a^{-6}.b^{-8}}}}} = ab\sqrt{\frac{f}{l}}$.

(6.) $\sqrt[m]{\sqrt[n]{\sqrt[p]{\left(\frac{a}{b}\right)^m \left(\frac{c}{d}\right)^n \left(\frac{e}{f}\right)^p \left(\frac{ac}{bd}\right)^p}}} = \sqrt[mp]{\frac{a}{b}}\sqrt[mp]{\frac{c}{d}}\sqrt[mn]{\frac{ace}{bdf}}$.

Greatest common measure of numbers.

Let us add some examples concerning the greatest common measure of numbers.

Given numbers: 544 416.

Ans. 32.

Given " 916 2201.

Ans. 31.

Given " 1261 1079.

Ans. 13.

Given " 1267 916.

Ans. 1.

CHAPTER II.

OPERATIONS ON POLYNOMIALS.

ARTICLE I.

Addition and Subtraction.

§ 58. ADDITION.—The addition of polynomials is substantially the same as that of monomials, since to add one polynomial to another is nothing more than to add a number of monomials to another number of monomials. And to add together several polynomials signifies to add together as many monomials as are those contained in the given polynomials. All, therefore, that has been said (18) with regard to the addition of monomials is applicable to the case of polynomials. Hence, the addition of the polynomials

$$\begin{array}{l} a^2 + 4ab + 3c^2 - d, \\ -3a^2 + ac - 3ab + l, \\ 7a^2 + ab - l + d, \end{array}$$

is obtained by writing in succession all their terms, each with its proper sign.

Reduction of similar terms.

However, before making this operation, and in order to obtain a simpler result, observe whether similar terms, (10,) or equal, are to be found in the polynomials. Because, all such terms are expressed by a single term similar to them, and having for coefficient the sum or the difference of the partial coefficients. For example, the first of the given polynomials contains the terms $+a^2$, $+4ab$; the second, the similar terms $-3a^2 - 3ab$, and the third polynomial, the terms $+7a^2 + ab$. Now, these six terms can be represented by only two equivalent to them, since the sum of the similar terms $a^2 - 3a^2 + 7a^2 = 5a^2$, and that of the terms $4ab - 3ab + ab = 2ab$. Again, the first polynomial contains the term $-d$, and the third the term $+d$, which are mutually elimi nated, as well as the terms $+l$ and $-l$, the first in the second, and the last in the third polynomial. Hence, the sum of the given polynomials is

$$5a^2+2ab+3c^2+ac.$$

Rule. And generally, *to obtain the sum of given polynomials, write, first, any of them as given, then a second, so that the similar terms shall fall under the corresponding terms of the first, and so all the other polynomials. Reduce then the similar terms, and annex those terms which are alone.*

Examples. § 59. Add together the polynomials

$$(1.)\quad\begin{cases}3a^2b+7b^2c-9c^2q-13q^2b,\\ -7a^2b+3c^2q+q^3b-la,\\ -8b^2c+6c^2q+3q^2b+3la.\end{cases}$$

Arranging these polynomials according to the preceding rule, we will have

$$\begin{array}{lrrrrr} & 3a^2b & +7b^2c & -9c^2q & -13q^2b & \\ & -7a^2b & & +3c^2q & +q^2b & -la \\ & & -8b^2c & +6c^2q & +2q^2b & +3la \\ \hline \text{Sum} & -4a^2b & -b^2c & & -10q^2b & +2la. \end{array}$$

Add together the polynomials

$$(2.)\quad\begin{cases}4a^3d+3c^2b-9m^2n,\\ 4m^2n+ab^2+5c^3b+7a^3d,\\ 6m^2n-5c^3b+4mn^2-8ab^2,\\ 7mn^2+6c^3b-5m^2n-6a^3b,\\ 7c^3b-10ab^2-8m^2n-10d^4,\\ 12a^3d-6ab^2+2c^3b+mn.\end{cases}$$

And also

$$(3.)\quad\begin{cases}a^3-b^3+3a^2b-5ab^2,\\ 3a^3-4a^2b+3b^3-3ab^2,\\ a^3+b^3+3a^2b,\\ 2a^3-4b^3-5ab^2,\\ 6a^2b+10ab^2,\\ -6a^3-7a^2b+4ab^2+2b^3.\end{cases}$$

Answers:

(2.) Sum, $17a^3d + 18c^3b - 12m^2n - 23ab^3 + 11mn^2 - 10d^4 + mn$.

(3.) Sum, $a^3 + b^3 + a^2b + ab^2$.

§ 60. SUBTRACTION.—To subtract a polynomial B from another polynomial A, means to find the *difference* between the two polynomials; that is, another polynomial D, which, if added to B, gives A for sum. Applying now to these expressions the reasoning made (21) with regard to simple monomials, we may easily infer, that

Rule. *The polynomial* B *is subtracted from* A *by adding to this a polynomial opposite to* B.

We do not need to prove that the polynomial opposite to B contains the same terms of B, but with opposite signs.

Examples. Take $B = 6m^3b - 5a^2b^2 + l$.

From $A = 3m^2b + 4m^3c - 6a^2b^2$.

The polynomial opposite to B is

$$-6m^3c + 5a^2b^2 - l.$$

Hence,
$$A - B = 3m^2b + 4m^3c - 6a^2b^2 \\ \underline{\qquad\qquad - 6m^3c + 5a^2b^2 - l.}$$

And
$$D = 3m^2b - 2m^3c - a^2b^2 - l.$$

Adding, in fact, this D to B, we will obtain A.

(1.) $\begin{cases} \text{Take } a + 2b + c + 5d \\ \text{from } 4a + 3b - 2c + 8d. \end{cases}$

(2.) $\begin{cases} \text{Take } -5ab + 7b^2 - 19a^2 + 2m \\ \text{from } 12ab + 3b^2 - 17a^2 + 3m. \end{cases}$

(3.) $\begin{cases} \text{Take } 10m^2b + 10m^3 - 10m^3b^2 \\ \text{from } 10m^3 + 4m^2b - 5m^3b^2. \end{cases}$

(4.) $\begin{cases} \text{Take } -a^2 + 2ab - b^2 \\ \text{from } a^2 + 2ab + b^2. \end{cases}$

(5.) $\begin{cases} \text{Take } -t^2yg - r^3s + 12mn \\ \text{from } 11r^3s + 12s^2 - 13t^2qg. \end{cases}$

Answers:

(1.) $D = 3a + b - 3c + 3d.$

(2.) $D = 17ab - 4b^3 + 2a^2 + m.$

(3.) $D = 5m^3b^2 - 6m^2b.$ (4.) $D = 2a^2 + 2b^2.$

(5.) $D = 12r^3s + 12s^2 - 12t^2qg + 12mn.$

ARTICLE II.

Multiplication and Division.

§ 62. MULTIPLICATION.—The multiplication of a polynomial A by another polynomial B, consists like that of monomials (24) in finding the product P, an expression either monomial or polynomial, whose numerical value is equal to the product of the numerical values of the factors A and B.

But multiplying A by each of the terms of B, and summing up all these partial products, the sum must be the product of A by B. Therefore, to obtain the product of a polynomial A by another polynomial B,

Rule. *Take the sum of the product of all the terms of* A *by each of the terms of* B, *or vice versa.*

Taking A for multiplicand, and B for multiplier, or *vice versa*, in both cases we will have the same terms to be added. The product of polynomials is usually indicated by enclosing them within parentheses, as follows: (A) (B).

Examples. § 63. Let $A = a - b$, $B = c - d$, we will have

$$A . B = (a - b)(c - d) = P.$$

And

$$(a - b)(c - d) = (a - b)c - (a - b)d$$
$$= ac - cb - ad + bd.$$

This example may be also used to demonstrate the correctness of the rule of signs.

Rule of signs demonstrated. In the preceding product we have followed the known rule of signs. But, making $a - b = m$, $c - d = n$, we can see with another process that the rule to be followed with regard to signs must be that already given, (25, 26.)

Add b to both members of the first equation, and d to both members of the second, we will have (16)

$$\left.\begin{array}{l} a = m + b \\ c = n + d \end{array}\right\} (o,)$$

and consequently, $a.c = (m + b)(n + d.)$.
The terms of these factors being all positive, their product is manifestly altogether positive; and consequently we will have

$$a.c = mn + bn + md + bd.$$

And since the difference $bd - bd = o$, the product $a.c$ will not be changed by adding this difference to it, and so it will be

$$a.c = mn + bn + md + bd + bd - bd.$$

Now the terms $bd + md$, having d for common factor, may be represented by $(b + m)d$, and likewise, the terms bd and bn, which have b for common factor, can be represented by $(d + n)b$. Hence, making the substitution, the preceding equation will become

$$a.c = (b + m)d + (d + n)b + mn - bd,$$

but the first (o) gives $(b + m) = a$, and the second $(d + n) = c$. Therefore, substituting again

$$a.c = a.d + c.b + m.n - bd.$$

Add now to both members of this equation the trinomial $bd - cb - ad$, we will have

$$a.c + b.d - c.b - a.d = m.n,$$

but m is $(a - b)$, and n is $(c - d)$; hence, from the last equation $(a - b)(c - d) = ac - cb - ad + bd$.
Exactly as we have obtained, following the rule: like signs give a positive, and unlike signs a negative product.

Remark on common factors. In the preceding demonstration we have mentioned and even separated the *common factor* from some terms. When a polynomial is multiplied by a monomial, this multiplier affects all the terms of the product likewise, and for this reason the multiplier is called common factor. And, *vice versa*, when the terms of a given polynomial are affected by the same factor, the polynomial can be considered as the product of another polynomial by that factor, and may be decomposed accordingly. For instance, from

$$P = am^2 + and + rsa,$$

we infer $$P = a(m^2 + nd + rs).$$

Nay, this decomposition can be performed with regard to a certain number only of terms, and when some of the terms of a polynomial are affected by one common factor, and some by another; partial decompositions can be evidently made: for example, from $P = ma^2b - nqd + ml + nrs$,

we will have $P = m(a^2b + l) + n(rs - qd)$

Examples of multiplication.

(1.) $\begin{cases} \text{Multiply } A = a^2 + 2ab + b^2 \\ \quad\text{by } B = a^2 - 2ab + b^2. \end{cases}$

Write first the partial products of A by each of the terms of B; take then their sum, or the product of P, as follows:

$$A.B = \underline{(a^2 + 2ab + b^2)}(a^2 - 2ab + b^2)$$

$$\begin{array}{llllll} \text{1st partial product...} & a^4 + 2a^3b + a^2b^2. & & & & \\ \text{2d partial product...} & & -2a^3b - 4a^2b^2 - 2ab^3. & & & \\ \text{3d partial product...} & & & a^2b^2 + 2ab^3 + b^4. & & \\ \hline \text{(1.)} \quad P = & a^4 & -2a^2b^2 & + b^4. & & \end{array}$$

(2.) $\begin{cases} \text{Multiply } A = a^3b^2 + 3a^2b^3 + 3ab^4 + b^5 \\ \quad\text{by } B = ab^2 - 4a^2b + 2a^3. \end{cases}$

(3.) $\begin{cases} \text{Multiply } A = 3a^2\sqrt{b} + 3m\sqrt[3]{c.a^2} - \dfrac{3m^2\sqrt[3]{c^2a^4}}{a^2\sqrt{b}} \\ \quad\text{by } B = \sqrt{b} - \dfrac{m}{a^2}\sqrt[3]{ca^2}. \end{cases}$

(4.) $\begin{cases} \text{Multiply together} \\ \quad (a - b),\ (b - c),\ (c - d),\ (d - e). \end{cases}$

(5.) $\begin{cases} \text{Multiply } A = a^2 + 2ab + b^2 \\ \quad\text{by } B = a - b. \end{cases}$

(6.) $\begin{cases} \text{Multiply } A = a + b \\ \quad\text{by } B = a - b \end{cases}$

Answers:

(2.) $P = 2a^6b^2 + 2a^5b^3 - 5a^4b^4 - 7a^3b^5 - a^2b^6 + ab^7.$

(3.) $P = 3a^2b - 6\frac{m^2}{a^2}\sqrt[3]{(ca^2)^2} + \frac{3m^3c}{a^2\sqrt{b}}.$

(4.) $P = abcd - b^2cd - ac^2d + bc^2d - abd^2 + b^2d^2 + acd^2 - bcd^2 - abce + b^2ce + ac^2e - bc^2e + abde - b^2de - acde + b^2de.$

Or else, $P = a[bd(c - e) + de(b - c)] + b^2[c(e - d) + d(d - e)] + c^2[a(e - d) - b(d - e)] + d[a(cd - bd) + b(cb - cd)].$

(5.) $P = a^3 + a^2b - ab^2 - b^3.$ (6.) $P = a^2 - b^2.$

The following examples deserve to be noticed on account of their frequent and useful applications.

(1.) $\begin{cases} \text{Multiply } A = 1 + z + z^2 + z^3 + \ldots + z^n \\ \qquad\quad \text{by } B = 1 - z. \end{cases}$

(2.) $\begin{cases} \text{Multiply } A = a + b\sqrt{-1} \\ \qquad\quad \text{by } B = a - b\sqrt{-1}. \end{cases}$

(3.) $\begin{cases} \text{Multiply } A = a + b\sqrt{-1} \\ \qquad\quad \text{by } B = h + k\sqrt{-1}. \end{cases}$

(4.) $\begin{cases} \text{Multiply } A = a - b\sqrt{-1} \\ \qquad\quad \text{by } B = h - k\sqrt{-1}. \end{cases}$

Observe, that the exponent n of the last term of A, in the first of these examples, is a whole number, containing one unity less than the number of terms of the same polynomial.

Answers:

(1.) $P = 1 - z^{n+1}.$ (2.) $P = a^2 + b^2.$

(3.) $P = ah - bk + (ak + bh)\sqrt{-1}.$

(4.) $P = ah - bk - (ak + bh)\sqrt{-1}.$

We may remark, with regard to the second of these products, that in changing the factors a into h, and b into k, we would have had

$$P = h^2 + k^2.$$

Singular property of numbers. § 64. We can demonstrate now a singular property of numbers, which is contained in the following theorem:

If two numbers M and N are such, that each one of them may be resolved into two square numbers, that is, $M = a^2 + b^2$, $N = h^2 + k^2$, the product M . N of the same number may likewise be resolved into two square numbers.

From the last example (2) and its equivalent we have

$$(a^2 + b^2)(h^2 + k^2) = [(a + b\sqrt{-1})(a - b)\sqrt{-1})][(h + k\sqrt{-1})(h - k)\sqrt{-1})]$$

$$= [(a + b\sqrt{-1})(h + k\sqrt{-1})][(a - b\sqrt{-1})(h - k\sqrt{-1})].$$

Again, from the last examples (3) and (4), we have

$$(a + b\sqrt{-1})(h + k\sqrt{-1}) = (ah - bk) + (ak + bh)\sqrt{-1}$$

$$(a - b\sqrt{-1})(h - k\sqrt{-1}) = (ah - bk) - (ak + bh)\sqrt{-1}.$$

Therefore, $(a^2 + b^2)(h^2 + k^2) = [(ah - bk) + (ak + bh)\sqrt{-1}][(ah - bk) - (ak + bh)\sqrt{-1}]$.

But we have seen in the preceding number that the product of the sum by the difference of any two quantities, is equal to the difference of the squares of the same quantities; hence, the product, or second member of the last equation, is

$$(ah - bk)^2 - ((ak + bk)\sqrt{-1})^2;$$

that is, $(a^2 + b^2)(h^2 + k^2) = (ah - bk)^2 - ((ak + bh)\sqrt{-1})^2$

But $((ak + bh)\sqrt{-1})^2 = (ak + bh)^2 . \sqrt{-1})^2 = -(ak + bh)^2$,

hence, $(a^2 + b^2)(h^2 + k^2) = (ah - bk)^2 + (ak + bh)^2$.

That is to say, the product of two numbers $M = (a^2 + b^2)$, $N = (h^2 + k^2)$, is equal to the sum of two square numbers.

For example, take M = 40 and N = 58, we will have

$$M = (6^2 + 2^2),\ N = (7^2 + 3^2,)$$

and consequently, M . N, or

$$40 . 58 = (6 . 7 - 2 . 3)^2 + (6 . 3 + 2 . 7)^2$$

$$= 36^2 + 32^2$$

$$= 1296 + 1024 = 2320.$$

§ 65. DIVISION.—To divide a polynomial A by another polynomial B, is to find a polynomial, or even monomial expression Q, the *quotient*, which, multiplied by the divisor B, gives for product the *dividend* A.

Polynomials arranged.

To obtain the quotient, it is expedient to arrange both dividend and divisor, according to the powers of the same letter; that is, writing, in the first place, the term in which the power of the letter is the highest, then the term where the power of the same letter is the highest of the remaining, and so on. This is the arrangement usually made, although it would be equally profitable to arrange the terms in the inverted order, commencing, namely, with the lowest power and increasing in order to the highest. Thus, for example, the polynomials

$$\begin{aligned} \mathrm{A} &= a^2 + 2ab + b^2, \\ \mathrm{B} &= 2a^3 + 2a^2b + ab^2 + b^3, \end{aligned}$$

are both arranged according to the decreasing powers of a, and according to the increasing powers of b.

When the polynomials are thus arranged, the operation of division is easily performed. But to see better on what principle this operation rests, let us multiply together the two preceding polynomials arranged A and B, marking each partial product, we will have

Operation explained.

$$\begin{array}{ll} & \underline{(a^2+2ab+b^2)}(2a^3+2a^2b+ab^2+b^3) \\ p' \ldots\ldots\ldots\ldots\ldots\ldots & 2a^5+4a^4b+2a^3b^2, \\ p'' \ldots\ldots\ldots\ldots\ldots\ldots & \qquad 2a^4b+4a^3b^2+2a^2b^3, \\ p''' \ldots\ldots\ldots\ldots\ldots & \qquad\qquad a^3b^2+2a^2b^3+ab^4, \\ p^{\text{iv}} \ldots\ldots\ldots\ldots\ldots\ldots & \underline{\qquad\qquad\qquad a^2b^3+2ab^4+b^5} \\ p'+p''+p'''+p^{\text{iv}} = & 2a^5+6a^4b+7a^3b^2+5a^2b^3+3ab^4+b^5 = \mathrm{P}. \end{array}$$

Let us now observe, first, that the product which results, is arranged according to the powers of the letters of the factors, and it is not difficult to see that it cannot be otherwise. Secondly, the first and the last term of the same product are merely produced by the multiplication of the first terms of the factors, and by the multiplication of the last terms. Hence, dividing the first term of the product by the first term of one of the factors, the quotient must be the first term of the other factor. Hence also, generally, when both dividend and divisor are arranged according to the powers of the same letter, the first term of the quotient is obtained by dividing the first term of the dividend by the first of the divisor.

Now, after having obtained the first term of the quotient, we can have also the partial product p' of the divisor by the quotient. For example, in the case before us dividing P by the polynomial A, we will obtain the first term of the quotient, that is, $2a^3$, by dividing the first term of P by the first of A. Now, multiplying A by $2a^3$, we obtain p', which subtracted from P, gives for remainder a polynomial $P = p'' + p''' + p^{iv}$; that is to say, the product of the same A by the remaining terms of the quotient. Repeating the operation with the divisor A and the dividend P′, we will obtain the next term of the quotient, which in our case is $2a^2b$, and, in consequence, the second partial product p'', and so on. Hence, the rule,

Rule. *To divide a polynomial* M *by another polynomial* N, *arrange, first, both according to the powers of the same letter. Then divide the first term of the dividend by the first of the divisor, and mark the quotient. Multiply then by this term the divisor* N, *and subtract the product from* M, *and taking the remainder for dividend, repeat the same operation until the end.*

	Divisor.	Dividend.	Quotient.
Examples.	$h + k)$	$h^2 + hk - hz - kz$	$(h - z$
1st product.......		$h^2 + hk$	
1st remainder....		$- hz - kz$	
2d product.......		$- hz - kz$	
2d remainder....		0	

The product may be taken with changed signs, and so the remainders may be obtained by simple addition.

When the dividend contains many terms, it is not necessary to write each time all those which belong to the remainder, but it is enough to write as many terms as there are in the divisor, as in the following example:

Divisor. Dividend. Quotient.

$$
\begin{array}{l}
a^2+2ab+b^2)a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7(a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \\
\qquad -a^7-2a^6b-\quad a^5b^2 \\
\hline
\qquad\qquad 5a^6b+20a^5b^2+35a^4b^3 \\
\qquad\qquad -5a^6b-10a^5b^2-\ 5a^4b^3 \\
\hline
\qquad\qquad\qquad 10a^5b^2+30a^4b^3+35a^3b^4 \\
\qquad\qquad\qquad -10a^5b^2-20a^4b^3-10a^3b^4 \\
\hline
\qquad\qquad\qquad\qquad 10a^4b^3+25a^3b^4+21a^2b^5 \\
\qquad\qquad\qquad\qquad -10a^4b^3-20a^3b^4-10a^2b^5 \\
\hline
\qquad\qquad\qquad\qquad\qquad 5a^3b^4+11a^2b^5+7ab^6 \\
\qquad\qquad\qquad\qquad\qquad -5a^3b^4-10a^2b^5-5ab^6 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad a^2b^5+2ab^6+b^7 \\
\qquad\qquad\qquad\qquad\qquad\qquad -a^2b^5-2ab^6-b^7 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0.
\end{array}
$$

In like manner, divide

(1.) $\begin{cases} A = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \\ \quad \text{by} \\ B = a^2 - 2ab + b^2. \end{cases}$

(2.) $\begin{cases} A = h^3 + k^3 \\ \quad \text{by} \\ B = h + k. \end{cases}$

(3.) $\begin{cases} A = a^5 - 7a^4b + 12a^3b^2 + a^2b^3 - 13ab^4 + 6b^5 \\ \quad \text{by} \\ B = a^3 - 4a^2b - 2ab^2 + 3b^3. \end{cases}$

(4.) $\begin{cases} A = x^4 + x^2y^2 + y^4 \\ \quad \text{by} \\ B = x^2 + xy + y^2. \end{cases}$

Answers:

(1.) $Q = a^2 - 2ab + b^2.$
(2.) $Q = h^2 - hk + k^2.$
(3.) $Q = a^2 - 3ab + 2b^2.$
(4.) $Q = x^2 - xy + y^2.$

Remark concerning the arrangement of polynomials.

The arrangement of polynomials according to the powers of the same letter, is not an indispensable requisite to obtain the quotient. This can be also obtained without such an arrangement. Nay, this must necessarily be done when the polynomials cannot be arranged. Such is, however, the nature of the process, that the same quotient will be obtained with different arrangements, although the disposition, the form of the terms, and their signs may appear different in the quotient for different arrangements.

For example, dividing the same polynomial A, differently arranged, as follows: $A = a^2 + 2ab + b^2$

$$= 2ab + a^2 + b^2$$

by $B = a + b.$

With the first arrrangement, we have

$$Q = a + b.$$

With the second, $Q = 2b + a - b$

$$= a + b.$$

So likewise, let the dividend be

$$A = 2abl + mnl + ql + 2abr + mnr + qr + 2abs + mns + qs.$$

And differently arranged,

$$A = mnr + qr + 2abr + mns + qs + 2abs + mnl + ql + 2abl.$$

Divide it by $B = l + r + s.$

With the first arrangement, we will find

$$Q = 2ab + mn + q.$$

With the second,

$$Q = \frac{mnr}{l} + \frac{qr}{l} + \frac{2abr}{l} + \frac{mns}{l} + \frac{qs}{l} + \frac{2abs}{l}$$
$$+ mn + q + 2ab$$
$$- \frac{mnr}{l} - \frac{mns}{l} - \frac{qr}{l} - \frac{qs}{l} - \frac{2abr}{l} - \frac{2abs}{l},$$

which is manifestly equal to

$$2ab + mn + q,$$

as for the first arrangement.

In order, however, to diminish useless labor, it is always expedient to arrange as much as possible the given polynomials. So, for example, the following dividend and divisor,

$$A = abcd + cdm + mn + abmn + m^2n + 3a^2b + 3abm$$
$$B = m + ab,$$

may be arranged in this manner:

$$A = 3a^2b + abcd + abmn + 3abm + cdm + m^2n + mn$$
$$B = ab + m.$$

And we will find

$$Q = 3ab + cd + mn + \frac{mn}{ab + m}.$$

The dividend not exactly divisible by the divisor.

§ 66. In this last example, after having found the first three terms of the quotient, the remainder to be divided is *mn*, which cannot be divided by B; we add, therefore, as the last term of the quotient, a fractional expression having the remainder for the numerator, and the divisor for the denominator.

This case takes place whenever the dividend is not the exact product of the divisor and the quotient. Let us give another example. Divide

$$A = h^3 - 3kh^2 + 3k^2h - 2k^3 \text{ by } B = h - k,$$

we will find $$Q = h^2 - 2kh + k^2 - \frac{k^3}{h-k}.$$

The correctness of the process to find out the quotient in all cases, can be demonstrated also in the following manner:

Call $t', t'', t''', \ldots$ the first, the second, the third term, &c. of the quotient obtained by dividing A by B, and call $r', r'', r''', \ldots$ the remainders corresponding to each term of the quotient, we will have

$$r' = A - t'B$$
$$r'' = A - t'B - t''B = A - (t' + t'')B$$
$$r''' = A - t'B - t''B - t'''B = A - (t' + t'' + t''')B, \text{ \&c.}$$

And $$r^{(n)} = A - B(t' + t'' + t''' + \ldots + t^{(n)}).$$

Suppose now, that after having obtained the n^{th} term of the quotient, we stop the operation. The polynomial

$$t' + t'' + t''' + \ldots + t^{(n)},$$

represents the quotient, and $r^{(n)}$ the last remainder. Now, from the preceding equation, we infer

$$A = B(t' + t'' + t''' + \ldots + t^{(n)}) + r^{(n)}.$$

But the quotient of A divided by B, must be such a quantity, which, if multiplied by B ought to give A, or, which is the same, $B(t' + t'' + t''' + \ldots + t^{(n)}) + r^{(n)}$, but this product is evidently obtained by multiplying $$(t' + t'' + t''' + \ldots + t^{(n)}) + \frac{r^{(n)}}{B}$$

by B. Therefore,

$$\frac{A}{B} = t' + t'' + t''' + \ldots + t^{(n)} + \frac{r^{(n)}}{B}.$$

These remarks, as we have observed, are applicable alike to all cases, whatever might be the form of the dividend and of the divisor, and even of the terms of the quotient.

It may occur that some, and even all the terms of the quotient have a fractional form. Dividing, for example,

$$A = a + a^2 + a^3 \text{ by } B = b + ab,$$

we will find $$Q=\frac{a}{b}+\frac{a^3}{b}-\frac{a^4}{b}+\frac{a^5}{b+ab}.$$

Number of the terms of the quotient indefinite. § 67. In this example, the remainder zero cannot be found, and consequently, continuing the operation, the number of the terms of the quotient becomes indefinite. And in cases similar to this we may add the remainder, with the divisor for denominator, either after the first, the second, the third term, &c. of the quotient. Let us see another example:

$$\begin{array}{lll}
\text{Divisor.} & \text{Dividend.} & \text{Quotient.} \\
h+k) & k^3 & \left(\frac{k^3}{h}-\frac{k^4}{h^2}+\frac{k^5}{h^3}-\right. \\
 & -k^3-\frac{k^4}{h} & \\
\cline{2-2}
 & \qquad -\frac{k^4}{h} & \\
 & \qquad +\frac{k^4}{h}+\frac{k^5}{h^2} & \\
\cline{2-2}
 & \qquad\qquad\quad +\frac{k^5}{h^2} & \\
 & \qquad\qquad\quad -\frac{k^5}{h^2}-\frac{k^6}{h^3} & \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad -\frac{k^6}{h^3}. &
\end{array}$$

Suppose now the operation to be interrupted after the first, after the second, and after the third term of the quotient, &c., we will have the following equivalent equations

$$\frac{k^3}{h+k}=\frac{k^3}{h}-\frac{\frac{k^4}{h}}{h+k}=\frac{k^3}{h}-\frac{k^3}{h+k}\cdot\frac{k}{h}.$$

$$\frac{k^3}{h+k}=\frac{k^3}{h}-\frac{k^4}{h^2}+\frac{k^5}{h+k}\cdot\frac{k^2}{h^2},$$

$$\frac{k^3}{h+k}=\frac{k^3}{h}-\frac{k^4}{h^2}+\frac{k^5}{h^3}-\frac{k^3}{h+k}\cdot\frac{k^3}{h^3};$$

and generally,

$$(a)\ \frac{k^3}{h+k}=\frac{k^3}{h}-\frac{k^4}{h^2}+\frac{k^5}{h^3}-\ldots\pm\frac{k^{n+2}}{h^n}\mp\frac{k^3}{h+k}\cdot\frac{k^n}{h^n}.$$

From the simple inspection of the order in which the signs of the preceding equations follow one another, and from the uniformity of the process to obtain any number of terms for the quotient, it is plain that the upper of the two signs placed before the last terms (a) is to be used when n is an uneven number, and the other when n is even.

With regard to the formula (a), three different cases can happen. That is, we may have $h = k$, or $h < k$, or $h > k$:

in the first case, $\frac{k}{h} = 1$;

in the second, $\frac{k}{h} > 1$;

in the third, $\frac{k}{h} < 1$.

In the last of these cases, by increasing indefinitely the number of the terms, and consequently, n in the formula (a), the factor $\frac{k^n}{h^n}$ of the last term will more and more approach to zero, and consequently likewise the whole term

$$\pm \frac{k^3}{h+k} \cdot \frac{k^n}{h^n} \ldots\ldots (r).$$

But if, by increasing the number of the terms in (a), the last of them constantly approaches to zero, we may say that the polynomial

$$\frac{k^3}{h} - \frac{k^4}{h^2} + \frac{k^5}{h^3} - \ldots \pm \frac{k^{n+2}}{h^n},$$

which contains all the terms of (a) with the exception of the last, is such that by increasing the number of its terms it constantly approaches to the determined value of the first number of (a), namely, to

$$\frac{k^3}{h+k} \ldots\ldots (s).$$

We call this fractional expression (s), to designate that it is equal to the *sum* of all the terms

$$(a') \ldots\ldots \frac{k^3}{h}, -\frac{k^4}{h^2}, \frac{k^5}{h^3}, -\frac{k^6}{h^4}, \frac{k^7}{h^5}, \ldots$$

of the *series* indefinitely protracted; that is,

$$(a'') \quad \frac{k^3}{h+k} = \frac{k^3}{h} - \frac{k^4}{h^2} + \frac{k^5}{h^3} - \ldots\ldots$$

The term (r) here omitted is called residual term, or *residuum*, after the n^{th} term.

In the other two cases of $\frac{k}{h} = 1$ or > 1, this residuum cannot be omitted, because when $\frac{k}{h} = 1$, (r) is constantly equal to $\pm \frac{k^3}{h+k}$, and when $\frac{k}{h} > 1$, the value of (r) constantly increases by adding units to n.

From the formula (a), dividing first both members by k^3, and making then $h = 1$, we deduce the two following useful equations:

$$(a''') \begin{cases} \dfrac{1}{h+k} = \dfrac{1}{h} - \dfrac{k}{h^2} + \dfrac{k^2}{h^3} - \dots \pm \dfrac{k^{n-1}}{h^n} \mp \dfrac{1}{h+k} \cdot \dfrac{k^n}{h^n} \\ \dfrac{1}{1+k} = 1 - k + k^2 - \dots \pm k^{n-1} \mp \dfrac{k^n}{h+k}. \end{cases}$$

The remarks made with regard to the general formula (a) are evidently applicable to the last two, and their residuums can be omitted whenever $\frac{k}{h} < 1$. With regard to the second of (a'''), the fraction $\frac{k}{h}$ cannot be < 1, unless k itself is < 1; that is, unless k be a fraction; because, since in that formula $h = 1$, the expression $\frac{k}{h}$ is nothing else but k. Supposing, therefore, k to be a fraction, we will have
$$\frac{1}{1+k} = 1 - k + k^2 - k^3 + \dots\dots$$
an indefinite series.

The quotient of the same form as the imaginary expressions divided by one another.

§ 68. When the imaginary expression $a + b\sqrt{-1}$ is divided by another imaginary expression $c + e\sqrt{-1}$, the quotient or fraction
$$\frac{a + b\sqrt{-1}}{c + e\sqrt{-1}},$$
remains unaltered when its numerator and denominator are multiplied by the same expression; for example, by $c - e\sqrt{-1}$. Effecting this multiplication, the numerator (68) becomes
$$ac + be + (cb - ae)\sqrt{-1};$$
and the denominator, $\qquad c^2 + e^2,$
and, consequently,
$$\frac{a + b\sqrt{-1}}{c + e\sqrt{-1}} = \frac{ac + be}{c^2 + e^2} + \frac{cb - ae}{c^2 + e^2}\sqrt{-1}.$$
Calling A the real quantity $\frac{ac + be}{c^2 + e^2}$, and B the coefficient $\frac{cb - ae}{c^2 + e^2}$, likewise real, of $\sqrt{-1}$, the same quotient will be represented by
$$A + B\sqrt{-1}.$$

That is to say, ***the quotient, no less than the product of two imaginary expressions of the form*** **$a+b\sqrt{-1}$,** ***is another imaginary expression of the same form.***

ARTICLE III.

Formation of Powers and Extraction of Roots.

§ 69. **POWERS.**—There is no difference between the formation of powers of monomials and that of polynomials, since the same operation, which in the former case is to be made about a monomial root, is to be made about a polynomial in the second. All, therefore, that concerns this operation with regard to monomials (Chap. I. 39) is to be applied to the case of polynomials. Hence, when the positive exponent m is a whole number, the operation to be performed to obtain the power of a given polynomial, or root A, is to multiply that same root by itself, as many times as there are units in m. Let us take, for example, the most simple case. That is, let the given root be the binomial $1+z$, and let us take in succession for exponent $m=2$, $m=3$, &c., we will have

Examples. (a)

$$\left\{\begin{array}{l}(1+z)^2=(1+z)(1+z)=1+2z+z^2\\ (1+z)^3=(1+z)^2(1+z)=1+3z+3z^2+z^3\\ (1+z)^4=(1+z)^3(1+z)=1+4z+6z^2+4z^3+z^4,\\ \qquad\qquad\&c.\end{array}\right.$$

Observe now, that

$$1+2z+z^2=1+2z+\frac{2(2-1)}{2}z^2$$

$$1+3z+3z^2+z^3=1+3z+\frac{3(3-1)}{2}z^2+\frac{3(3-1)(3-2)}{2.3}z^3$$

$$1+4z+6z^2+4z^3+z^4=1+4z+\frac{4(4-1)}{2}z^2+\frac{4(4-1)(4-2)}{2.3}z^3$$
$$+\frac{4(4-1)(4-2)(4-3)}{2.3.4}z^4.$$

Therefore,

$$(1+z)^2=1+2z+\frac{2(2-1)}{2}z^2$$

$$(1+z)^3=1+3z+\frac{3(3-1)}{2}z^2+\frac{3(3-1)(3-2)}{2.3}z^3$$

$$(1+z)^4=1+4z+\frac{4(4-1)}{2}z^2+\frac{4(4-1)(4-2)}{2.3}z^3$$
$$+\frac{4(4-1)(4-2)(4-3)}{2.3.4}z^4, \text{ \&c.}$$

From the uniformity observable in these evolutions of the binomial $1+z$, we could infer by analogy a general law, extending alike to all positive and whole powers, so that the m^{th} power of $(1+z)$ would be given by the formula

$$(a')\ldots(1+z)^m=1+mz+\frac{m(m-1)}{2}z^2+\frac{m(m-1)(m-2)}{2.3}z^3$$
$$+\frac{m(m-1)(m-2)(m-3)}{2.3.4}z^4+\ldots,$$

containing $(m+1)$ terms in the evolution, because, were they $m+2$, the last of them would have among its factors $(m-m)$, which renders the whole term equal to zero. The same factor $(m-m)$ would be found also in all the following terms. Therefore, the said evolution of $(1+z)^m$ cannot contain more than $m+1$ terms, and consequently, three terms when $m=2$, and four when $m=3$, &c., as we have seen in the preceding examples.

It is moreover plain, that in the same evolution of $(1+z)^m$, the highest and last exponent of z is equal to m.

Newtonian formula.

Let us now make $z=\frac{y}{x}$, and, substituting this value in (a'), we will have

$$\left(1+\frac{y}{x}\right)^m=1+m\frac{y}{x}+\frac{m(m-1)}{2}\frac{y^2}{x^2}+\frac{m(m-1)(m-2)}{2.3}\frac{y^3}{x^3}+\ldots$$

but $\left(1+\frac{y}{x}\right)^m=\frac{(x+y)^m}{x^m}$, multiplying then both members of the equation by x^m, we will find

$$(e)\ldots(x+y)^m = x^m + mx^{m-1}y + \frac{m(m-1)}{2}x^{m-2}y^2 + \frac{m(m-1)(m-2)}{2.3}x^{m-3}y^3 + \ldots$$

The form and the order in which the terms of the binomial $(x+y)^m$ evolved, succeed one another, was first discovered by Sir Isaac Newton; hence, this formula (e) bears the name of its discoverer.

By mere induction, however, the correctness of the evolution is not demonstrated. A rigorous demonstration of it may be seen in the following number. Let us now observe, first, that the formula

General term.

$$(e')\ldots\frac{m(m-1)(m-2)\ldots(m-(p-1))}{1.2.3.4\ldots p}x^{m-p}y^p,$$

represents the $(p+1)^{\text{th}}$ term of (e). Substituting, in fact, 1, 2, 3, 4, &c. instead of p, we will find the second, the third, the fourth term, &c. of the evolution. Hence (e') is called the *general term* of the series.

$$mx^{m-1}y,\ \frac{m(m-1)}{2}x^{m-2}y^2,\ \frac{m(m-1)(m-2)}{2.3}x^{m-3}y^3.\ \ldots.$$

Secondly, substituting 2 and 3, instead of m, in the formula (e), we have

$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

Just as we would obtain from the first and second (a), substituting in them $\frac{x}{y}$ instead of z. Hence, *the square of a binomial is given by the square of the first term, the double product of the first by the second term, and the square of the second term.*

Square and cube of any binomial.

The cube of a binomial contains the cube of the first term; the triple product of the square of the first by the simple *second term, the triple product of the simple first term by the square of the second, and finally, the cube of the second term.*

Binomial theorem demonstrated.

§ 70. Let us now resume the formula (a'). We say that when m is a whole and positive number, the evolution of $(1+z)^m$ is exactly represented by the second member of (a').

Observe, first, that $(a') \times (1+z)$ or

$$\left(1 + mz + \frac{m(m-1)}{2}(z^2 + \ldots\right)(1+z)$$

$$= 1 + mz + \frac{m(m-1)}{2}z^2 + \frac{m(m-1)(m-2)}{2.3}z^3 + \ldots$$

$$+ \quad z + \quad m \quad z^2 + \frac{m(m-1)}{2}z^3 + \ldots$$

And, adding together the similar terms,

$$(a') + (1+z) = 1 + (m+1)z + \left(\frac{m-1}{2}z + 1\right)mz^2 + \left(\frac{m-2}{3} + 1\right)\frac{m(m-1)}{2}z^3 + \ldots$$

$$= 1 + (m+1)z + \frac{(m+1)m}{2}z^2 + \frac{(m+1)m(m-1)}{2.3}z^3 + \ldots$$

That is to say, $(a') + (1+z)$ is equal to a polynomial which contains $m+2$ terms, one term more than those of (a'), and this is easily proved in the same manner as we have demonstrated that the number of terms in (a') do not exceed $(m+1)$.

Now, substituting $m+1$ instead of m in (a'), we have

$$1 + (m+1)z + \frac{(m+1)m}{2}z^2 + \frac{(m+1)m(m-1)}{2.3}z^3 + \ldots$$

that is, the product of (a') by $(1+z)$. Therefore, whatever be the whole and positive number m, the product of (a') by $(1+z)$ is obtained by changing in (a') m into $m+1$. We may now proceed to demonstrate the theorem as follows:

If the polynomial (a') is the exact evolution, for example, of the third power of $1+z$ when $m = 3$, it will be also the evolution of the fourth power of the same binomial, making $m = 4$; but if (a') is the exact evolution of the fourth power of $1+z$ when $m = 4$, it must be also the evolution of the fifth power of $1+z$, making $m = 5$, and so on. Therefore, whatever be the number of units in m above 3, provided with $m = 3$, the polynomial (a') gives the evolution of

$(1+z)^3$ it will give the evolution also of $(1+z)^m$, so that this power in our supposition is exactly represented by the polynomial (a'). But, indeed, (a') is the evolution of $(1+z)^2$, $(1+z)^3$, when we make $m=2$, $m=3$, which may be easily verified, comparing (a') with the formulas (a). Therefore, when m is any whole and positive number, the evolution of $(1+z)^m$ is given by the polynomial (a'). And since we have seen, that making in (a') $z=\frac{y}{x}$, the same (a') is changed into the Newtonian formula (e); the evolution, therefore, of the binomial $(x+y)^m$, according to the known law, is also rigorously demonstrated, whatever may be the whole and positive value of m.

Singular property of the coefficients of the formula (e).

§ 71. The coefficients of the formula (e) have the singular property of representing the numbers of combinations of m different quantities. Hence, before we speak of the extraction of roots, we will dwell here upon this subject, as well as upon the use of the preceding formulas, in order to find out some numerical properties.

Permutations.

Let $a, b, c, d, \ldots\ldots$ represent m different symbols. The first a taken in succession with each one of the following $b, c, d, \ldots$ will give us $(m-1)$ *binaries* $ab, ac, ad, \ldots.$ But, joining in equal manner the second symbol b with all the others, we will have again $(m-1)$ *binaries;* and let the same be said of the third, fourth, and so on. In this manner we will have a joined to b, and again b joined to a, a joined to c, and also c to a, &c.; for this reason this arrangement of symbols is termed *permutation.* But for each symbol the number of permutations of the letters taken *two* and *two* is $m-1$, and the symbols are m in number; therefore, the whole number of permutations of m letters taken two and two is

$$m(m-1).$$

Again, the binary ab joined in succession with each one of the remaining symbols, will give us $m-2$ *ternaries;* and the same we must say of all the other binaries. Now the number of permutations of m symbols taken two and two is $m(m-1)$; therefore, the number of permutations of m symbols taken *three* and *three,* is

$$m(m-1)(m-2).$$

It is now easy to see, that the number of permutations of m symbols taken *four* and *four,* is

$$m(m-1)(m-2)(m-3),$$

and generally, the number of permutations of m symbols taken p and p, is

$$m(m-1)(m-2)\ldots.(m-(p-1)).$$

But if to the preceding binaries, ternaries, &c., we would add all those which come out from the repetition of the same symbols—for example, aa, bb, ... aaa, aab, ... bbc, ... to the number $m(m-1)$ of the terms taken two and two—we must then add m more binaries; and since

$$m(m-1)+m=m^2,$$

the number of permutations of m terms taken two and two, with the repetition of the same symbol, is

$$m^2.$$

We would find in a similar manner, that m^3 gives the number of permutations with the repetition of the same symbols of the m terms taken three and three. But let us investigate the subject, assuming it in a more general point of view.

Permutations with repetitions.

§ 72. Suppose m letters to be taken $p-1$, and $p-1$, in all possible manners, without excluding the repetition of the same terms.

To obtain the same m symbols taken p and p, it is enough to add in succession the m terms to each of the collections of the same terms taken $p-1$ and $p-1$, and make the addition in the last place.

To demonstrate this proposition, let f be one of the m symbols. Among the terms taken $(p-1)$ and $(p-1)$, there must be some collections in which f does not enter at all, others in which f enters only once, others in which it enters twice, &c. The same must be with regard to the symbols taken p and p; but all those collections of terms taken p and p, and excluding f, must certainly terminate with any symbol except f. To obtain, therefore, all the collections of terms taken p and p, with the exclusion of f, it is enough to add in succession to each one of those collections, taken $p-1$ and $p-1$, and which exclude f, all the m symbols except f; but adding to the same collections also f, we will obtain all those taken p and p, containing f once, and in the last place.

After the collections of symbols taken $p-1$ and $p-1$, and excluding f, come those which contain f only once, some of which must have f for the first, some for the second, some for the third term, and so on. Now, adding to the end of each one of them the m terms, one after another with the exclusion of f, we will evidently obtain all the possible collections of m terms taken p and p, and in which f enters only once, either in the first, second, or third place, and so on, except those which contain f in the last place; but we have seen how they are obtained in the first addition, and now if to the same terms take $p-1$

and $p-1$, and containing f only once, we add f once more for the last term, we will have those collections of terms taken p and p, in which f enters twice, with one of them, however, constantly at the end.

If now to each collection of terms taken $p-1$ and $p-1$, and containing f twice, we add in succession all the m terms with the exception of f, these, together with the last mentioned, will give us all the collections of terms taken p and p in which f enters twice in all possible ways. It is now plain, that adding, likewise, to all the remaining collections of terms taken $p-1$ and $p-1$ all the m letters in succession, we will obtain all those taken p and p, when f enters three, four, five times, &c., in all possible ways. But whatever is demonstrated with regard to the symbol f is evidently applicable to all the others. Hence, adding in succession the m terms to each collection of the same terms taken $p-1$ and $p-1$ with all kinds of permutations and repetitions, we will obtain all the same permutations with the repetitions of the terms taken p and p; and the number of the collections of the terms taken p and p is evidently m times as great as the number of collections of the terms taken $p-1$ and $p-1$.

Let us now call N the number of collections of the m terms taken $p-1$ and $p-1$. The number of collections of the same terms taken p and p will be given by $\mathrm{N} \,.\, m$.

But supposing $p = 3$, and, consequently, $p-1 = 2$; as we have seen in the preceding number N in this case is $= m^2$. Therefore, the number of permutations with repetitions of m terms taken three and three is

$$m^2 \,.\, m = m^3.$$

But if the number of collections of m terms taken three and three is $\mathrm{N} = m^3$, it follows, likewise, that the number of collections of the same terms taken four and four must be $m^3 \,.\, m = m^4$, and so on; and consequently, we generally infer that the number of permutations, with repetitions of m terms taken p and p, is

$$\mathrm{N} = m^p.$$

How the same process gives the general formulas of permutations without repetitions.

§ 78. We are now able to infer again the general formula of simple permutations. Suppose the permutations of m symbols taken $p-1$ and $p-1$. Each symbol enters only once in these collections, and the symbol f, for example, in some of them will be the first, in others the second, in others the third, and so on, till the last. To all these f cannot be added to obtain the permutations of symbols taken p and p; but adding other symbols we will obtain the terms taken p and p containing f in the first, second, and third place, and so on till the last, exclusively. Hence, to all the other permutations of terms taken $p-1$ and $p-1$, and excluding f, this symbol cannot be added except at the end. The same thing ought to be said of any other symbol: if the symbol is already in the permutation containing $p-1$ terms, it is not to be added; if it is not in it, it must be added only at the end.

Now each permutation contains by supposition $p-1$ symbols. The number, therefore, of symbols to be successively added at the end of each one of them is $m-(p-1)$,

and so we will obtain all the permutations of terms taken p and p. So that, calling ν the number of permutations of m terms taken $p-1$ and $p-1$, for the number of simple permutations of the same m terms taken p and p, we will have

$$\nu[m-(p-1)].$$

Let us take for example $p=3$, and, consequently, $p-1=2$. In this case we have seen (71), that $\nu=m(m-1)$; therefore, the number of simple permutations of m symbols taken three and three is

$$m(m-1)(m-2).$$

And if the number ν of permutations of m symbols taken three and three is $m(m-1)(m-2)$, that of the same symbols taken four and four is

$$m(m-1)(m-2)(m-3),$$

and so on. And generally, the number of permutations of m terms taken p and p, is that already found (71), with another process—

$$m(m-1)(m-2)\ldots\ldots(m-(p-1)).$$

Corollary. Taking $p=m$, we will have the number of permutations which may be obtained by the collection of all the m symbols. But observe, that in this case the last factor $(m-(p-1))$ of the general formula becomes $(m-m+1)=1$, and, consequently, the factor before the last is 2, and the preceding one 3, &c. Hence, the number of simple permutations which can be formed with all the m symbols is

$$m(m-1)(m-2)\ldots\ldots3.2.1.$$

This same formula gives us, besides, the number of permutations which can be obtained from the same two, the same three, and the same p letters. So that, if we wish to know, for example, how many permutations are made by the first five of the m symbols taken five and five, it is enough to substitute 5 instead of m in the preceding formula. Calling now, ν_2, ν_3, ... ν_p, the number of permutations formed with any two and the same symbols; with any three .... with any p, we will have

$$r_2 = 2\,.\,1 \qquad\qquad = \quad 2$$
$$r_3 = 3\,.\,2\,.\,1 \;\ldots\ldots\; = 2\,.\,3$$
$$r_p = p(p-1)\ldots.\,2\,.\,1 = 2\,.\,3\ldots.\,(p-1)p.$$

It is now easy for us to determine the simple *combinations* of m given symbols.

Combinations.

§ 74. In the simple *combinations* we exclude all the collections of terms in which at least one symbol is not different from the symbols of another collection. For example, the symbols a and b can be combined with c, d, ... but after having taken abc, abd, we exclude all the permutations which can be formed with the same terms a, b, c, or a, b, d.

Call now n_3 the number of simple combinations of terms taken three and three. From that which we have just observed, and from the preceding formula r_3, it follows first, that the number of permutations of any three and the same symbols is $2\,.\,3$, and, consequently,

$$n_3 + (2\,.\,3)$$

is the number of permutations of all the m terms taken three and three. But this same number is expressed also by $m(m-1)(m-2)$; therefore,
$$n_3 + (2\,.\,3) = m(m-1)(m-2),$$
and, consequently,
$$n_3 = \frac{m(m-1)(m-2)}{2\,.\,3}.$$

In equal manner, calling n_p the number of combinations made with m symbols taken p and p, and multipyling n_p by r_p we will obtain the number of corresponding permutations $n_p + (2\,.\,3\ldots p)$, but the same number is also given by $m(m-1)\ldots.\,(m-(p-1))$; therefore,

$$n_p + (2\,.\,3\ldots.\,p = m(m-1)\ldots(m-(p-1)),$$

and, consequently.

$$n_p = \frac{m(m-1)(m-2)\ldots(m-(p-1))}{2\,.\,3\,.\,4\ldots.\,p}.$$

Now, this is the general formula of the coefficients of (e), and making in it $p = 2$, $= 3$, $= 4$, $=$, &c., we obtain the coefficients of the third, of the fourth term, and so on, of the same evolution. But at the same time, making $p = 2$, $= 3$, $=$, &c., we have the numbers of combinations of m terms taken two and two, three and three, and so on; therefore, the coefficient of the third term of the evolution of the binomial $(x+y)^m$ gives the number of combinations of m symbols taken two and two, the coefficient of the fourth term gives the number of combinations of m terms taken three and three, and so on.

On some property of numbers.

§ 75. Let us now make use of the formula (e) to find a certain property of numbers.

The formula (e) may be changed indefinitely by giving different values to y, to x, and to m. Let us take instead of the exponent m, the number k prime in itself and make $x = h$ and $y = 1$; the evolution (e) will become

$$(h+1)^k = h^k + kh^{k-1} + \frac{k(k-1)}{2}h^{k-2} + \frac{k(k-1)(k-2)}{2.3}h^{k-3} +, \&c.$$
$$+ \frac{k(k-1)(k-2)\dots 3.2}{2.3.4\dots(k-1)}h + 1;$$

and consequently,

$$(h+1)^k - h^k - 1 = kh^{k-1} + \frac{k(k-1)}{2}h^{k-2} + \dots\dots$$
$$+ \frac{k(k-1)\dots 3.2}{2.3\dots(k-1)}h.$$

We may now pass to demonstrate the following theorem:

Theorem.

If the whole number N *is not exactly divisible by the prime number* k, *this number* k *will certainly exactly divide* $N^{(k-1)} - 1$.

All the terms of the second member of the last equation contain the factor k; the whole member, therefore, is exactly divisible by k, and, consequently, also the first member $(h+1)k - h^k - 1$ in which h may have any numerical value. Make, therefore, successively $h = 1$, $= 2$, $= 3$, &c., the trinomial will become successively

$$2^k - 1 - 1,\ 3^k - 2^k - 1,\ 4^k - 3^k - 1, \&c.,$$

and always exactly divisible by k, and, consequently, the sum of the first two, or three, or four, and so on, will also be divisible by k. Now, the first of these trinomials is equivalent to

$$2^k - 2;$$

and consequently the sum of the two first is

$$3^k - 3,$$

and the sum of the three first

$$4^k - 4,$$

and so on. So that N being any whole number, the binomial

$$N^k - N$$

is exactly divisible by k. But

$$N^k - N = N(N^{k-1} - 1).$$

Hence, whatever be the whole number N, the product $N(N^{k-1} - 1)$ is always exactly divisible by the prime number k. Consequently if k does not exactly divide N, it must necessarily (53) divide $N^{k-1}-1$.

Evolution of the binomial when the exponent is negative.

§ 76. We had (67) the following equation:

$$\frac{1}{1+z}=1-z+z^2-z^3+z^4-\ldots.$$

where z must have a fractional numerical value, that is, less than unity.

But change in (a') (69) m into -1, we will have

$$(1+z)^{-1}=1-z+z^2-z^3+z^4-\ldots.$$

with the second member indefinite. Now, $(1+z)^{-1}=\frac{1}{1+z}$ and $\frac{1}{1+z}$ is really equal to the indefinite series, provided the numerical value of z be a fraction; the formula (a'), therefore, besides the evolution of the binomial $1+z$ raised to any whole and positive power m, gives also the indefinite series equivalent to $(1+z)^{-1}$, changing m into -1, with the condition, however, that the numerical value of z be less than unity.

Nay, more, the numerical value of z being such, changing in the formula (a') the sign to m, it will give us an indefinite series, and equivalent to any whole and negative power of $(1+z)$. Because, from

$$(1+z)^{-1}=1-z+z^2-z^3+z^4-z^5+\ldots.$$

we have, also

$$(1+z)^{-m}=(1-z+z^2-z^3+z^4-z^5+\ldots)^m.$$

Call S, for the sake of brevity, all the terms of the indefinite series, with the exception of the first, the preceding formula will become

$$(1+z)^{-m}=(1+S)^m.$$

Now,

$$(1+S)^m=1+mS+\frac{m(m-1)}{2}S^2+\frac{m(m-1)(m-2)}{2.3}S^3+\ldots$$

And with

$$S=-z+z^2-z^3+\ldots=-z(1-z+z^2-\ldots).$$

we have, also

$$S^2=z^2(1-z+z^2-\ldots)^2=z^2[1-2(z-z^2+\ldots)+(z-z^2+\ldots)^2]$$
$$S^3=-z^3(1-z+z^2-\ldots)^3=-z^3[1-3(z-z^2+\ldots)+\ldots]$$
$$S^4=z^4(1-z+z^2-\ldots)^4=z^4[1-4(z-z^2+\ldots)+\ldots], \text{ \&c.}$$

And, consequently,

$$mS=-mz(1-z+z^2-z^3+\ldots)$$

$$\frac{m(m-1)}{2}S^2=\frac{m(m-1)}{2}z^2(1-2z+3z^2-4z^3+\ldots)$$

$$\frac{m(m-1)(m-2)}{2.3}S^3 = -\frac{m(m-1)(m-2)}{2.3}z^3(1-3z\ldots), \text{ \&c.}$$

Or else,

$$mS = -mz + mz^2 - mz^3 + \ldots$$

$$\frac{m(m-1)}{2}S^2 = \frac{m(m-1)}{2}z^2 - m(m-1)z^3 + \ldots$$

$$\frac{m(m-1)(m-2)}{2.3}S^3 = -\frac{m(m-1)(m-2)}{2.3}z^3 + \ldots$$

And therefore,

$$1 + mS + \frac{m(m-1)}{2}S^2 + \frac{m(m-1)(m-2)}{2.3}S^3 + \ldots$$

$$= 1 - mz + \left[m + \frac{m(m-1)}{2}\right]z^2$$

$$- \left[m + m(m-1) + \frac{m(m-1)(m-2)}{2.3}\right]z^3 + \ldots$$

$$= 1 - mz + \frac{m(m+1)}{2}z^2 - \frac{m(m+1)(m+2)}{2.3}z^3 + \ldots$$

That is, $(1+S)^m$, or its equivalent

$$(1+z)^{-m} = 1 - mz + \frac{m(m+1)}{2}z^2 - \frac{m(m+1)(m+2)}{2.3}z^3 + \ldots$$

which is precisely the formula immediately obtained from (a') changing m into $-m$.

Taking now $z = \frac{y}{x}$ with $x > y$, this value may be substituted in the last formula, which will become

$$(x+y)^{-m} = x^m - mx^{m-1}y + \frac{m(m+1)}{2}x^{m-2}y^2 - \frac{m(m-1)(m-2)}{2.3}x^{m-3}y^3 + \ldots$$

which is the binomial formula extended to the case of the negative exponent, in which, however, the first term x of the binomial must be greater than the second y.

But $x + y = y + x$, and, consequently, $(x + y^{-m} = (y + x)^{-m})$. Arranging, therefore, the terms of the evolution as above, and in such a manner that the greater of the given binomial be the first, the above formula is applicable to all cases without exception.

The extraction of roots of polynomials, the same as the extraction of roots of monomials.

§ 77. EXTRACTION OF ROOTS.—The m^{th} root of a polynomial P is another polynomial R, which raised to the m power gives P. To find out the *root* R, is to extract the m^{th} root of the polynomial

P, m is called *index* or degree, and the radical sign used is the same as for monomials (47).

The process of the operation is to be inferred from the opposite one of raising to powers, which may be done in two different ways, either examining the most general case, and thence deriving practical rules for particular and determined cases, or commencing with the simplest case. The first method is unquestionably superior to the other. But the second, beside being easier, affords us all that which may conveniently find a place in the present article.

The root of a polynomial is another polynomial.

And it is first to be observed, that the square or cubical root, or more generally the m^{th} root of a polynomial, must necessarily be another polynomial; because a polynomial raised to any power preserves constantly a monomial form, and therefore, in the equation

$$\sqrt{P} = R;$$

R must be at least a binomial, for example, $a + b$; then P, which is the square of R, is equal to the product $(a + b)(a + b) = a^2 + 2ab + b^2$; that is, the square of the first term of R, plus the double product of its two terms, plus the square of the last term.

In this supposition, therefore, P must be a trinomial, and one of its terms is the square of the first term of R. Hence, taking the square root of this term, we will have the first of R. The two remaining terms of P are the double product of the first by the second term of R, plus the square of the second. Now, dividing the double product by the double of the first term of R already obtained, we will evidently obtain the second and last terms of R. This process will be better understood with an example.

Example.

Let the given polynomial be

$$P = m^2r^6 + q^4 + 2mr^3q^2.$$

In order to have the square of one of the terms of R in the first place, and the double product in the second, arrange the polynomial according to the powers of a letter. Thus, we will have

$$\begin{array}{rl|l}
 & \text{P} & \text{R} \\
\hline
 & m^2r^6 + 2mr^3q^2 + q^4 & mr^3 + q^2 \\
 & -m^2r^6 & \\
\hline
r_{,} = & 2mr^3q^2 + q^4 & \\
 & -2mr^3q^2 - q^4 & \\
\hline
r_{,,} = & 0, &
\end{array}$$

for the square of the first term of P is mr^3, and, consequently mr^3 is the first term of R. Subtract now the square of mr^3 from P, the remainder $r_{,}$ contains the double product of mr^3 by the other term of R to be found for the first term. Divide then this term by the double of mr^3, which gives q^2 for quotient, the second term of R. Subtracting now from $r_{,}$ the product of the second term just found by the double of the first, plus the product of the second term by itself, the second remainder $r_{,,}$ must be equal to zero, if P is really the square of the binomial R, as it is in the present example.

But the polynomial P, although a perfect square, will not always be the square of a binomial. Still, whatever might be the number of terms in R, the process of the operation to derive R from P is always the same.

The process to find the square root of any polynomial is always the same.

Let, in fact, the root R be a polynomial composed of any number n of terms $a + b + \ldots + x \ldots$ We will have then $\quad P = (a + b + c + \ldots + x.)^2$

and $\quad \sqrt{P} = R = a + b + c + \ldots\ldots$

Calling now A_1 the same polynomial R, with the exception of the first term, and A_2 the same polynomial with the exception of the two first terms, and so on; and calling t_2 the two first terms $a + b$, t_3 the first three $a + b + c$, and so on, besides

$$R = a + b + c + \ldots\ldots$$

we will have, also,

$$R = a + A_1$$
$$R = t_2 + A_2$$
$$R = t_3 + A_3, \text{ \&c.}$$

And consequently since $P = R^2$,

$$P = (a + A_1)^2 = (t_2 + A_2)^2 = (t_3 + A_3)^2 = \ldots$$

or, which is the same,

$$\begin{aligned} P &= a^2 + 2aA_1 + A_1^2 \\ &= t_2^2 + 2t_2A_2 + A_2^2 \\ &= t_3^2 + 2t_3A_3 + A_3^2 \\ &= \ldots\ldots \\ &= t_{n-1}^2 + 2t_{n-1}x + x^2, \end{aligned}$$

in which last member, t_{n-1} represents all the terms of R, with the exception of the last, and x the n^{th} or last term of R. Supposing, moreover, the whole number p to be any number between 2 and $n-1$ inclusively, the general expression equivalent to P, will be

$$P = t_p^2 + 2t_pA_p + A_p^2 \qquad (g),$$

and, consequently,

$$\sqrt{P} = t_p + A_p.$$

Now, $t_p = t_{p-1} + r$, t_p, namely, commencing with the first term of R, contains one term r more than t_{p-1}, but from $t_p = t_{p-1} + r$, we have, also

$$t_p^2 = t_{p-1}^2 + 2t_{p-1}r + r^2,$$

which value, substituted in (g), gives

$$P = t_{p-1}^2 + 2t_{p-1}r + r^2 + 2t_pA_p + A_p^2,$$

but in (g) we may change at pleasure p into $p-1$, in which case

$$P = t_{p-1}^2 + 2t_{p-1}A_{p-1} + A_{p-1}^2.$$

Hence, taking the second member of this and of the preceding equation, we will have another equation, as follows:

$$t_{p-1}^2 + 2t_{p-1}A_{p-1} + A_{p-1}^2 = t_{p-1}^2 + 2t_{p-1}r + r^2 + 2t_pA_p + A_p^2,$$

from which, taking t_{p-1}^2, which is in both members, we will have

$$2t_{p-1}A_{p-1} + A_{p-1}^2 = 2t_{p-1}r + r^2 + 2t_pA_p + A_p^2,$$

in which it is to be observed that r is the first of the terms of A_{p-1}.

Substituting now in this formula, instead of p, the numbers 2, 3, 4, ... $n-1$ in succession, observe, that $t_1 = a$, the first term of R, and $A_{n-1} = x$, the last term of the same R, we will have:

$$\begin{aligned} 2aA_1 + A_1^2 &= 2ab + b^2 + [2t_2A_2 + A_2^2] \\ 2t_2A_2 + A_2^2 &= 2t_2c + c^2 + [2t_3A_3 + A_3^2] \\ 2t_3A_3 + A_3^2 &= 2t_3d + d^2 + [2t_4A_4 + A_4^2] \\ &\&c. \\ 2t_{n-1}x + x^2 &= 2t_{n-1}x + x^2 + 0. \end{aligned}$$

That is, the last binomial within the brackets of the first equation is the first member of the second equation; the last binomial within the brackets of the second equation is the first member of the third equation, and so on. Making, therefore, a continual substitution, we will have

$$2a\mathrm{A}_1 + \mathrm{A}_1^2 = 2ab + b^2 + (2t_2c + c^2) + (2t_3d + d^2) + \ldots\ldots + (2t_{n-1}x + x^2).$$

Now, $\mathrm{P} = a^2 + 2a\mathrm{A}_1 + \mathrm{A}_1^2$; therefore,

$$(f) \quad \mathrm{P} = (a^2 + 2ab + b^2) + (2t_2c + c^2) + (2t_3d + d^2) + \ldots\ldots + (2t_{n-1}x + x^2).$$

Observe, that in the supposition of the polynomial R arranged according to the powers of a letter, the two first terms of P must contain the two highest powers of the same letter. Again, a being the first term of R, is also the first term of t_2, of t_3, &c. Hence a enters as factor in all the following terms $2t_2c$, $2t_3d$, ... and since, in the supposition of the polynomial $\mathrm{R} = a + b + c + d + \ldots$ arranged according to the powers of the same letter, this power diminishes gradually in c, in d, &c. So, also, after the trinomial $a^2 + 2ab + b^2$, the highest power of the letter is in the first term of the product $2t_2c$, and after the binomial $(2t_2c + c^2)$ the highest power of the same letter is to be found in the first term of the product $2t_3d$, &c.

General rule for the extraction of square roots of polynomials.

We may now proceed to give a general rule for the extraction of the square root of any polynomial P.

Arrange the polynomial according to the decreasing powers of a letter. The square root of the first term of the polynomial thus arranged will be the first term a of R, then subtract a^2 from P, and divide the second term of P and the first of the remainder r_1 by the double of the term already obtained, namely, by $2a$; the quotient will be the second term b of the root. Now multiplying b by the double of the first term and by itself, and subtracting it from r_1, the remainder r_2 will contain the terms $[2t_2c + c^2] + \ldots$. in which t_2, t_3 ... are again resolvable into other terms, as we have seen. Now divide the first term of this remainder r_2 by $2a$, the quotient must be the third term c of the root; multiplying now by c the double of the two preceding terms, that is, $2t_2$ and c itself, and subtracting this product from r_2, the next remainder r_3 will contain the terms $[2t_3d + d^2] + \ldots$.; and following constantly the same process, we will manifestly obtain all the terms of the root. The same rule can also be compendiously expressed as follows:

Rule for practice.

Take the square root of the first term of the arranged polynomial P, *and subtract the same* *first term from* P. *To obtain the following terms, divide the first terms of all the remainders by the double of the first term of the root. The first remainder is the given polynomial* P, *less its first term; the successive remainders are obtained, by subtracting from the preceding one the double product of the term of the root last obtained by all the preceding, plus the square of the same last term.*

Examples. Let the given polynomial be

$$P = a^6b^2 + 4a^5b^3 + ba^4b^4 + 4a^3b^5 + a^2b^6,$$

we will have

$$\begin{array}{rl} & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad R \\ & a^6b^2 + 4a^5b^3 + 6a^4b^4 + 4a^3b^5 + a^2b^6 \big(a^3b + 2a^2b^2 + ab^3 \\ & -a^6b^2 \\ \hline r_1 = & \qquad 4a^5b^3 + 6a^4b^4 + 4a^3b^5 + a^2b^6 \\ & -4a^5b^3 - 4a^4b^4 \\ \hline r_2 = & \qquad\qquad 2a^4b^4 + 4a^3b^5 + a^2b^6 \\ & \qquad\qquad -2a^4b^4 - 4a^3b^5 - a^2b^6 \\ \hline r_3 = & \qquad\qquad\qquad 0, \end{array}$$

So, also, from the given polynomials,

(1.) $P = a^6b^4 + 4a^5b^3 + 6a^4b^2 + 4a^3b + 2a^3b^2m + a^2 + 4a^2bm + 2am + m^2.$

(2.) $P = x^8y^2 + 2x^7y^3 + 3x^6y^4 + 2x^5y^5 + x^4y^6.$

(3.) $P = 4a^2b^2 - 12a^3b^3 + 13a^2b^4 - 6ab^5 + b^6.$

(4.) $P = 49m^4 - 84m^2n^2 + 36n^4,$

we will find

(1.) $R = a^3b^2 + 2a^2b + a + m.$

(2.) $R = x^4y + x^3y^2 + x^2y^3.$

(3.) $R = 2a^2b - 3ab^2 + b^3.$

(4.) $R = 7m^2 - 6n^2.$

Extraction of square roots of numbers. § 78. The practical rule given in arithmetic to extract square roots of numbers, contains the same process of operation as for polynomials, and we may demonstrate that it must be the same, although in some respects, apparently different in its application. The rule is as follows:

Practical rules: First. *Separate the given number into periods of two figures, each beginning with units and tens.* Second. *Extract then the square root of the last period, thus obtained, whether it contains two figures like the others or only one, and if this period is not a perfect square, take the root of the greatest square number contained in it.* Third. *Subtract then this same square number from the said period, and annexing to the remainder the first figure of the next period, divide the whole number by the double of the root obtained. The quotient will be the second figure of the root.* Fourth. *Annex now to the remainder the second figure also of the next period, and subtract from the whole the product obtained by multiplying by the second figure, the first figure of the root redoubled, with the second annexed to it.* Fifth. *Annex now to the second remainder, the first figure of the following period, and divide the number by the double of the root already obtained; the quotient will be the third figure.* Sixth. *After this annex the second figure of the period to the same remainder, and subtract from the whole number the product, which will be obtained by multiplying the two first figures of the root redoubled with the third annexed to them, and repeat upon this and the following remainders the same operation as above.*

Examples. The same rule will be better understood by an example. Let the given number be

$$N = 15539364,$$

which will be, separated into periods, as follows:

$$N = 15,\ 53,\ 93,\ 64.$$

Now the period 15 is not a perfect square, but 9 is the greatest square number contained in it, having 3 for its root. And according to the rule, we will have

$$\begin{array}{lll}
 & & \mathrm{N} \qquad\qquad\quad \mathrm{R} \\
 & & 15,53,93,64(3942 \\
 & \underline{3 \times 2} & \underline{-9} \\
\text{1st divisor,} & 6 \ldots\ldots\ldots & 65'3 \\
 & \underline{39 \times 2} & -\underline{621}\ldots(=-69 \times 9) \\
\text{2d divisor,} & 78 \ldots\ldots\ldots & 329'3 \\
 & \underline{394 \times 2} & -\underline{3136}\ldots\ldots(=-784 \times 4) \\
\text{3d divisor,} & 788 \ldots\ldots\ldots & 1576'4 \\
 & & -\underline{1576\,4}\ldots\ldots(=-7882 \times 2).
\end{array}$$

So, likewise, from the given numbers:

(1.) N = 539725824.
(2.) N = 567009.
(3.) N = 127449
(4.) N = 56821444,

we will find,

(1.) R = 23232.
(2.) R = 753.
(3.) R = 357.
(4.) R = 7538.

How the rule given for the extraction of square roots of polynomials is applicable to the extraction of square roots of numbers.

§ 79. But let us see more clearly how the process of the operation to obtain the square root of a number, is the same as that to be followed in the extraction of square roots of polynomials. And for this object let us remark, that

First remark. A compound whole number can always take the form of a polynomial.

1st. *A polynomial form can always be given to any compound whole number.*

Simple whole numbers.

We call here a compound number any whole number which contains more than one figure; as for instance, all the N's and R's of the preceding examples; and a simple number, a whole number of only one figure, as 2, 3, 9. Let now A be any simple number; the product

$$A \,.\, 10^{(m-1)}$$

will be a compound number, containing m figures, of which, commencing to reckon from units, A is the m^{th}, and the only one different from zero. Let, likewise, each of the symbols B, C, D, .. M,N represent a simple number, or even the cipher zero. The compound number $\nu = \mathrm{ABCD} \ldots\ldots \mathrm{MN}$,

which we suppose to contain m figures, and in which N represents units, M, tens, &c., can always take the form of a polynomial in the following manner:

$$\mathrm{A} . 10^{m-1} + \mathrm{B}10^{m-2} + \mathrm{C}10^{m-3} + \ldots\ldots + \mathrm{M} . 10 + \mathrm{N}.$$

Because, as $\mathrm{A} . 10^{m-1}$ contains $m - 1$ zeroes after A, so, also $\mathrm{B} . 10^{m-2}$ contains $m - 2$ zeroes after B, and $\mathrm{C} . 10^{m-3}$ contains $m - 3$ such zeroes after C, and so on. Therefore,

$$\begin{array}{cr} \multicolumn{2}{c}{\mathrm{A} . 10^{m-1} + \mathrm{B} . 10^{m-2} + \mathrm{C}10^{m-3} \ldots + \mathrm{M} . 10 + \mathrm{N}} \\ = & \mathrm{A0000} \ldots 00 \\ + & \mathrm{B000} \ldots 00 \\ + & \mathrm{C00} \ldots 00 \\ + & \mathrm{D0} \ldots 00 \\ + & \ldots\ldots \\ + & \mathrm{M0} \\ + & \mathrm{N} \\ \hline = & \mathrm{ABCD} \ldots \mathrm{MN} = \nu. \end{array}$$

Second remark. The square of a compound number cannot contain more figures than the double of the figures in the root.

Let us now observe,

2d. *The square of* ν *can neither contain more than* $2m$ *figures, nor less than* $2m - 1$.

In the present supposition, ν contains m figures. Now, 10^{m-1} is the *minimum* among the numbers, which, like ν, contain m figures, as 10^m is the *minimum* among compound numbers of $m + 1$ figures. Therefore, ν cannot be less than 10^{m-1}, and must be less than 10^m; hence, also ν^2 cannot be less than $(10^{m-1})^2 = 10^{2m-2}$, and must be less than $(10^m)^2 = 10^{2m}$. But 10^{2m-2} is the *minimum* among the numbers which contain $2m - 1$ figures; ν^2, therefore, which cannot be less than 10^{2m-2}, must contain at least $2m - 1$ figures. Again, 10^{2m} is the *minimum* among the numbers which contain $2m + 1$ figures; ν^2, therefore, which is less than this *minimum*, cannot contain more than $2m$ figures. Now, a number containing $2m$ figures may be separated into m periods of two figures each, and a number containing $2m - 1$ figures may, likewise, be separated into m periods, each of two figures, with the exception of the last, which cannot be of more than one figure. In both cases, however, the number of periods of ν^2 is the same as the number of figures in the root ν. Hence, when a square number is given, by separating the ciphers into periods of two figures each, we may immediately know what is the number of figures in the corresponding root. It remains now for us to see how these figures of the root may be found out from the periods in succession, through the same process with which the terms of the root are found out from the square polynomial.

Polynomial expression of the square of a number.

Let us first observe, that the first period commencing with units has no ciphers after itself; the second has two, and the third, four figures after them, and, generally the m^{th} period has $2m-2$ figures after itself. Representing now by $p_1, p_2, p_3, \ldots\ldots p_{m-1}, p_m$, these periods, the square of r will be evidently expressed by

$$(f') \quad r^2 = p_m 10^{2m-2} + p_{m-1}10^{2m-4} + \ldots\ldots + p_3 10^4 + p_2 10^2 + p.$$

But $\quad r^2 = [A.10^{m-1} + B.10^{m-2} + \ldots\ldots + M.10 + N]^2;$

and (77) the square of any polynomial is expressed by the formula (f) from which, in our case, we have

$$r^2 = A^2.10^{2m-2} + 2AB.10^{2m-3} + B^2.10^{2m-4} + (2t_2C.10^{m-3} + C^2.10^{2m-6})$$
$$+ (2t_3D.10^{m-4} + D^2.10^{2m-8}) + \ldots\ldots + (2t_{m-1}N + N^2),$$

where $\quad t_2 = A.10^{m-1} + B.10^{m-2}$

$$t_3 = A.10^{m-1} + B.10^{m-2} + C.10^{m-3}, \text{ \&c.};$$

and, consequently,

$$2t_2.C.10^{m-3} = 2AC.10^{2m-4} + 2BC.10^{2m-5}$$
$$2t_3D.10^{m-4} = 2AD.10^{2m-5} + 2BD.10^{2m-6} + 2CD.10^{2m-7}, \text{ \&c.}$$

Substituting now these values in the preceding formula, we will have

$$\begin{aligned}(f'')\, r^2 = A^2.10^{2m-2} + 2AB.10^{2m-3} + B^2.10^{2m-4}&\\ +2AC.10^{2m-4} + 2BC.10^{2m-5} + C^2.10^{2m-6}&\\ +2AD.10^{2m-5} + 2BD10^{2m-6}&\\ +, \text{ \&c.}&\end{aligned}$$

Both formulas (f') and (f'') give the same value of r^2 with a difference, however, which is to be remarked here. Suppose, for example, $A = 3$ and $B = 4$. The first and second terms of the square of r given by (f''), taken separately from the rest, will give us

$$\begin{aligned}A^2.10^{2m-2} + 2AB.10^{2m-3} &= 9.10^{2m-2} + 24.10^{2m-3}\\ &= 9.10^{2m-2} + (2.10 + 4)10^{2m-3}\\ &= 9.10^{2m-2} + 2.10^{2m-2} + 4.10^{2m-3}\\ &= 11.10^{2m-2} + 4.10^{2m-3}\end{aligned}$$

It appears from this reduction, that some units of the second term $2AB . 10^{2m-3}$ join the units of a higher order, and enter into the first term. The same thing occurs with regard to the following terms, and the difference between the formulas (f'), (f'') is that in the first the periods p_m, p_{m-1}, ... contain all the units which may possibly be reduced to their respective order; in the second, some of the units of a higher order form part of the successive terms. But when any square number is given, its periods are as in (f'), in which the terms of the root do not so distinctly appear as in (f''). We may still safely say, that $A^2 . 10^{2m-2}$, or the square of the first term of the root, is altogether included in the period p_m, and the double product $2AB . 10^{2m-3}$ of the first and second term of the root does not go beyond the first figure of the following period p_{m-1}, and the square $B^2 . 10^{2m-4}$ of the second term of the root does not go beyond the same period p_{m-1}, and so on. Let us observe, also, that comparing, for instance, the number

$$2AB . 10^{2m-3} + B^2 . 10^{2m-4},$$

or, which is the same, $(2A . 10 + B) B . 10^{2m-4}$,

with $$(2A . 10 + B)B,$$

the only difference to be found between the two numbers is that in the first, the product $(2A . 10 + B)B$ is followed by $2m - 4$ ciphers; in the second, the same product is followed by no cipher. In the case, therefore, in which the ciphers indicated by 10^{2m-4} would not be taken into account, the number $(2A . 10 + B)B$ may be used instead of $(2A . 10 + B)B . 10^{2m-4}$.

We may now proceed to see the reason of the operation to be performed and expressed by the rule, in order to extract the square root of any given number.

We must commence by taking for the first figure of the root the square root of the last period, or rather the square root of the highest square number contained in the period. Then, after having subtracted the same square number from the said period, we will obtain the second figure, by dividing by the double of the number already obtained for the root the remainder as far as the first figure, inclusively of the following period p_{m-1}; since the second figure of the root, multiplied by the double of the first, is within these limits. In this manner we have the part of the root which is obtained and expressed in separate terms by $A . 10 + B$, although, rigorously speaking, the same terms should be expressed by $A . 10^{m-1} + B . 10^m$; since A is the m^{th}, and B the

$(m-1)^{\text{th}}$ figure after units in the root; but in (f'') we have the terms $2AB . 10^{2m-3} + B^2 . 10^{2m-4}$, or their equivalent $(2A . 10 + B)B . 10^{2m-4}$, which are contained within the period p_{m-1} of (f'), and having no consideration for the following ciphers, as we do with regard to the figures in the root; this is the product of the double of the first cipher A, obtained for the root, plus the second cipher B, multiplied by B, which being subtracted from the remainder, as far as the whole period p_{m-1}, will give for the next remainder $2AC . 10^{2m-4} + 2BC . 10^{2m-5} +$, &c., or $2(A . 10 + B)C . 10^{2m-5} +$, &c. Hence, taking this remainder as far as the first figure of the period p_{m-2}, we have in it the product of the double of the root already obtained, multiplied by the third figure to be yet found; to find, therefore, this third cipher, we divide the remainder as far as the said figure of the period p_{m-2} by the double of the root obtained. The number then thus far obtained for the root is expressed by $A . 10^2 + B . 10 + c$. But taking the whole period p_{m-2} in the remainder, we have in it $2AC . 10^{2m-4} + 2BC . 10^{2m-5} + c^2 . 10^{2m-6}$, or $(2(A . 10^2 + B . 10) + C)C . 10^{2m-6}$, and having no consideration for the following ciphers expressed by 10^{2m-6}, the said remainder contains the product of the double of the two first figures of the root, plus the third figure, all multiplied by the same third figure; and this product being subtracted from the remainder will give us another remainder, which taken as far as the first figure of the following period p_{m-3}, contains the product $2(A . 10^2 + B . 10 + C)D . 10^{2m-7}$, which, having no consideration for the following ciphers, is the product of the double of the root obtained by the figure to be next found. Dividing, therefore, the last remainder as far as the said limit by the double of the root obtained, we will have for quotient, the fourth figure, &c.

It is not necessary to go on farther to see on what principles the rule given to extract square roots of numbers rests, and to see also, that the process of the operation is identically the same as that of the extraction of roots of polynomials, although somewhat more complicated, on account of the units of the squares of each figure of the root, separated in different orders and periods.

Extraction of cubical roots of polynomials.

§ 80. In the same manner as the rule to extract the square root from polynomials, is inferred from the formation of the square power, the rule to extract the cubical root of a given polynomial is deduced from the formation of the corresponding power.

Cubical power of a binomial, and of any polynomial.

Now the cubical power of a binomial contains (69) the cube of the first term, the triple product of the square of the first by the simple second term, the triple of the product of the square of the second by the simple first term, and finally, the cube of the second term. But a polynomial can be at pleasure divided into two parts, considering each part as a single term. In this manner the evolution of the cubical power of a binomial becomes applicable to any polynomial, and the rule to extract the cubical root inferred from the formation of the cubical power of a binomial is likewise generally applicable to the extraction of a cubical root of any polynomial. This will be better seen with an example. Let the given polynomial be

Examples.

$$\mathrm{P} = a^6 + 3a^5b + 3a^4b^2 + a^3b^3 + 3a^2b^2c + 3abc^2 + c^3 + 3a^4c + 6a^3bc + 3a^2c^2;$$

which, arranged according to the powers of a, gives us

$$\begin{array}{ll}
\mathrm{P}\left\{\begin{array}{l} a^6+3a^5b+3a^4b^2+a^3b^3+3a^2b^2c+3abc^2+c^3 \\ \qquad +3a^4c+6a^3bc+3a^2c^2 \end{array}\right. & \Big(a^2+ab+c \Big\} \mathrm{R} \\
-a^6 & \\
\hline
(r_1)\left\{\begin{array}{l} 3a^5b+3a^4b^2+a^3b^3+3a^2b^2c+3abc^2+c^3 \\ \qquad +3a^4c+6a^3bc+3a^2c^2 \end{array}\right. & \\
-3a^5b-3a^4b^2-a^3b^3 & \\
\hline
(r_2)\left\{\begin{array}{l} 3a^4c+6a^3bc+3a^2b^2c+3abc^2+c^3 \\ \qquad +3a^2c^2 \end{array}\right. & \\
-3a^4c-6a^3bc-3a^2b^2c-3abc^2-c^3 & \\
\qquad -3a^2c^2 & \\
\hline
(r_3) \qquad 0 &
\end{array}$$

The operation proceeds as follows: We extract the cubical root of the first term a^6, which root is a^2, and a^2 is the first term of R, namely, the first term of the root of the given polynomial. Taking then from P the cube of a^2, we will have the first remainder (r_1). Dividing now the first of (r_1) by $3a^4$, that is, by the triple product of the square of a^2, the quotient ab hence resulting is the second term of R. Taking then from (r_1) the triple product of the square of a^2 by ab, plus the triple product of the square of ab by a^2, plus the cube of ab, the remainder (r_2) resulting from this operation is equal to P, minus the cube of the root (a^2+ab) so far obtained. Therefore, dividing (r_2) by the triple product of the square of (a^2+ab), the first term resulting from this division is another term of R. Now the triple product of the square of (a^2+ab) is $3a^4+6a^3b+3a^2b^2$; and performing the division, we find c for the third term of R. Subtracting now from (r_2) the triple product of $(a^2+ab)^2$ by c, plus the triple product of (a^2+ab) by c^2, plus c^3 the remainder (r_3) will be equal to P, minus the cube of (a^2+ab+c), but (r_3) is found equal to zero; therefore, the cubical root of the given polynomial P is $R=a^2+ab+c$. The process of the operation is wholly founded in this, that the part of the root obtained is regarded as a single term, and the part to be obtained as a second term.

In similar manner, from

$$P = m^6+3m^5n+6m^4n^2+7m^3n^3+6m^2n^4+3mn^5+n^6,$$

we will find $\sqrt[3]{P}$, or

$$R = m^2+mn+n^2.$$

And from $P = 8a^6+12a^5b+6a^4b^2+a^3b^3$,

we will find $\sqrt[3]{P}$, or $R = 2a^2+ab$.

Extraction of cubical roots of numbers.

§ 81. The same process is applicable to the extraction of cubical roots of numbers. And recalling to mind that which we have already remarked (79) with regard to the extraction of square roots of numbers, it will be easy to see the identity of the operation with an example.

But let us first observe, that the cubical power N of a number r of m figures ABC.... cannot contain more than $3m$ figures, nor less than $3m-2$; therefore, dividing the given power N into periods of figures taken three and three, the last of these periods will either contain three figures like the others, or only two, or even one; secondly, the number of periods of N will be equal to that of the figures of its cubical root r. By observations similar to those previously made (79), we find, besides, that the cubical power of the first figure A of r is entirely in the last period of N, and the triple product of A^2 by B does not go beyond the first figure of the following period, and the cubical power of $(10A + B)$ is entirely within the same period, &c But let us see an example :

Let the given power or number N be 34012224, or separating it into periods, let

$$N = 34{,}012{,}224,$$

r or $\sqrt[3]{N}$ must then contain three figures, and we can represent it by

$$r = ABC.$$

The operation to find out these ciphers proceeds as follows:

$$\begin{array}{llr}
 & & N \qquad r \\
A = 3 & & 34{,}012{,}224(324 \\
A^3 = 27 & & \underline{-27} \\
3A^2 = 3\,.\,9 = 27,\ B = 2 & (r_1) & 7\ 0/12 \\
3(10^2A)^2B + 3(10A)B^2 + B^3 = 5768 & & \underline{-5\ 7\ 68} \\
3(10A + B)^2 = 3(32)^2 = 2187, & (r_2) & 1\ 2\ \ 442/24 \\
 & & \underline{-1\ 2\ \ 442\ \ 24} \\
\qquad C = 4 & (r_3) & 0. \\
\end{array}$$

$$3(10^2A + 10B)^2\,.\,c + 3(10^2A + 10B)c^2 + c^3 = 1244224.$$

The highest cubical power contained in the period 34 is 27, and the corresponding cubical root is 3, therefore, $A = 3$, and $A^3 = 27$, which being subtracted from the last period of N, we have the first remainder 7; to this remainder we join the first figure 0 of the following period and divide 70 by $3A^2$, that is, by the triple of the square of the root obtained; the quotient is 2, and therefore, the second figure B of r is 2. Now, joining to 70 the two remaining figures of the period 012, subtract from 7012 the triple product of the square of 30 multiplied by 2, plus the triple product of 30 multiplied by the square of 2, plus the cube of 2, that is, subtract 5768 from 7012. Add then to the remainder the first figure of the next period, and divide it by the triple product of the square of the root 32, obtained, namely, by 2187 · we find 4 for the quotient, and then repeating the operation as above, we will find zero for the last remainder. And applying this rule to other cases, we will find from $N = 1879080904$,

$$\sqrt[3]{N} = R = 1234;$$

from

$$N = 658503,$$

$$\sqrt[3]{N} = R = 87, \text{ \&c.}$$

SECOND PART.

ALGEBRAIC THEORIES.

CHAPTER I.

EQUATIONS.

Division of the chapter.

§ 82. THE present chapter will be divided into four articles: The first treating of the equations of the first degree; the second, of the equations of the second degree; the third, of some general properties of determined equations; and the last, of the resolution of the equations of the third and fourth degree.

The limits within which this treatise must be necessarily confined do not allow us to dwell much on this subject, which contains one of the best parts of analysis. We will treat briefly each article, without, however, leaving untouched such discussions as may afford a sufficient idea of the theory in question. But, first, let us here elucidate the general deduction inferred in the introductory article (16), namely, that the members of an equation equally modified form other equations.

Let a given equation be

$$mx+\frac{h}{n}-q=r-t \ldots\ldots (a).$$

Taking from both members the same quantity or quantities, the remainders will form another or other equations. Let us subtract $\frac{h}{n}-q$ from both members, we will have

$$mx+\frac{h}{n}-q-\frac{h}{n}+q=r-t-\frac{h}{n}+q;$$

that is, $$mx = r - t - \frac{h}{n} + q \quad (b).$$

Let us subtract from the same (a) $r - t$, we will have

$$mx + \frac{h}{n} - q - r + t = r - t - r + t,$$

that is, $$mx + \frac{h}{n} - q - r + t = 0 \quad (b').$$

Practical rules.

Comparing (b) and (b') with (a), we see that the terms $\frac{h}{n}$ and q of the first member of (a) are in the second member of (b) with changed signs, and the terms r and t of the second member of (a) are in the first member of (b') likewise with changed signs. We infer, therefore, this practical and general rule

First rule.

The terms of an equation can be transposed from one to another member, without destroying the equality, provided their signs be changed.

Dividing or multiplying both members of (a) by the same quantity, the products or quotients will form another equation, and dividing first both members by m, we will have

$$\frac{mx}{m} + \frac{h}{nm} - \frac{q}{m} = \frac{r}{m} - \frac{t}{m};$$

that is $$x + \frac{h}{nm} - \frac{q}{m} = \frac{r}{m} - \frac{t}{m} \quad (c).$$

Multiply now both members of (a) by n, we will have

$$mnx + \frac{nh}{n} - nq = nr - nt;$$

that is, $$mnx + h - nq = nr - nt \ldots (c').$$

Comparing now (c) and (c') with (a), it is easy to see that the coefficient m of the term mx in (a) is not to be found in the corresponding term of (c), but it divides all the others. Again, the denominator n of the term $\frac{h}{n}$ in (a) is not to be found in the corresponding term of (c'), but it is a common factor of all the other terms. Hence,

Second rule. *Any term of a given equation may be cleared of its coefficient by dividing all the other terms by the same coefficient.*

Third rule. *Any term of a given equation may be cleared of its denominator by multiplying all the other terms by the same denominator;*

The last of these rules is contained in the first, when the denominator is considered as the denominator of the coefficient of the term to which it belongs.

Examples. Applying now the preceding rules to the following examples:—

(1.) $nx^2 + q = m - b,$

(2.) $\frac{y}{m} = p - f,$

(3.) $\frac{n}{m}x^3 + g = l - h,$

we will have

(1.) $x^2 = \frac{m - q - b}{n}.$

(2.) $y = mp - mf = m(p - f).$

(3.) $x^3 = \frac{m(l - g - h)}{n}.$

Known and unknown quantities. § 83. Equations commonly contain known or given quantities, and unknown quantities or quantities to be found: the known quantities are generally expressed (20) by the first letters $a, b, c, \ldots$ of the alphabet, the unknown quantities by the last .... v, x, y, z. Hence, in the equation

$$ax^2 + b - z = c,$$

we would consider a, b, and c as given quantities, x and z as quantities to be determined.

Resolution of equations. The determination of these unknown quantities is called the resolution of the equation; thus, for instance, to find out $\frac{20}{3}$ or the value of x, which makes the first member of the equation

$$3x - 4 = 16,$$

equal to the second, is to resolve the same equation.

Determinate and indeterminate equations.

Now, when an equation contains only one unknown quantity, it is called a *determinate* equation; when it contains more than one unknown quantity, it is called *indeterminate*. The reason of such an appellation is, that an equation which contains only one unknown quantity, has either only one or a determinate number of resolutions; and an equation which contains more than one unknown quantity, has no determinate number of resolutions. The value of the unknown quantities are termed also *roots* of the equation.

Roots of equations.

ARTICLE I.

Equations of the First Degree.

Degree of the equations.

§ 84. THE degree of an equation is given by the highest exponent of the unknown quantity or quantities. Thus, for example, the equation

$$x^2 - ax = b,$$

in which the highest exponent of the unknown quantity x is 2, is an equation of the second degree, and the equation

$$y^3 - x^2 + ayx = m + y,$$

in which the highest exponent of the unknown quantities x and y is 3, is an equation of the third degree.

Equations, therefore, of the first degree are all those in which the exponents of the unknown quantities do not surpass unity; such, for instance, are the equations

$$ax + b = q, \ ay + bx = q - x, \text{ \&c.};$$

and generally,

$$ax^n - by^n + abx^{n-1} - qy^{n-2} + f = r,$$

is an equation of the n^{th} degree.

General formula of determinate equations of the first degree.

§ 85. Any determinate equation of the first degree may always be reduced to the simple form

$$x = A \quad (i),$$

because it cannot contain other terms except known quantities, and those in which the only unknown quantity x is either alone or affected by a coefficient; as, for instance, in the equation

$$ax + b - c + dx = \frac{1}{2}x - l + f.$$

Now we may first transpose all the known terms to the second, and the remaining to the first member, and have the equation

$$ax + dx - \frac{1}{2}x = c - b - l + f,$$

and again, $\left(a + d - \frac{1}{2}\right)x = c - b - l + f.$

The known terms $\left(a + d - \frac{1}{2}\right)$ can easily be reduced to a single term C, and likewise the terms $-b + c - l + f$, to the single term K, so that the same equation can simply be written as follows: $Cx = K$.

The unknown x being cleared of its coefficient, we will have

$$x = \frac{K}{C},$$

an equation of the same form as (i).

Resolution of any determinate equation of the first degree.

Now (i) is a resolved equation. Therefore, to resolve any determinate equation of the first degree,

Rule.

Transpose all the known terms to the second member, and all the others to the first; reduce each member to a single term, and clear the unknown quantity of its coefficient.

Examples. Resolve the equations

(1.) $2x - 3 + \frac{1}{4}x = 6.$

(2.) $7y - 7 = 13y - 26.$

(3.) $2z - 4 + \frac{1}{2} = \frac{7}{2} - 4z.$

(4.) $\frac{1}{2}x + 12 = \frac{3}{4}x$

(5.) $ay + b - c = my - f.$

Answers:

(1.) $x = 4.$ (2.) $y = \frac{19}{6}.$

(3.) $z = \frac{7}{6}.$ (4.) $x = 48.$

(5.) $y = \frac{c - b - f}{a - m}.$

Problems. § 86. Equations can be profitably used in the resolution of problems, since the conditions of any problem can be expressed by one or more equations. Let us see some examples:

First. What number is that which first multiplied by 2, and then divided by 7, gives 13 for the difference between the product and the quotient?

Ans. The product is $2x$, the quotient is $\frac{x}{7}$, the difference is $2x - \frac{x}{7}$; therefore, the equation is

$$2x - \frac{x}{7} = 13,$$

which, resolved, gives ... $x = 7$. That is, 7 is the number required.

Second. There is a certain number of apples to be distributed among another number of boys. If we give to each boy three apples, there are nine wanting; but if we give only two to each, then there are two remaining. How many are the apples, and how many the boys?

It is plain that after having found the number x of the apples, the number of boys is also found; because, dividing x by two, we have the number of boys, plus one; that is, the number of boys is
$$\frac{x}{2}-1;$$
the unknown quantity of the problem is then only x.

But adding 9 to x, and dividing $x+9$ by 3, we have again, according to the condition of the problem, the number of boys; that is,
$$\frac{x+9}{3}=\frac{x}{2}-1.$$
An equation which, resolved, gives
$$x=24 \ldots \text{number of apples};$$
and consequently,
$$\frac{x}{2}-1=11, \text{ number of boys}$$

Third. What is the number, which multiplied by 3 and divided by 7, gives a product and quotient whose difference is 20? Ans. $x=7$.

Fourth. Find such a number that the sum of one-third, one-sixth, and one-twelfth of it shall be equal to 21. Ans. $x=36$.

Fifth. A soldier receives every day twelve cents; but when he is engaged in the service, the first time in the month he receives twice as much; the second time three times as much; and the remaining days of service four times as much. At the end of a month of 30 days he receives five dollars and four cents. How many times was he engaged in the service? Ans. $x=5$.

Indeterminate equations of the first degree.

§ 87. The preceding examples and problems show that the value of the unknown quantity in determinate equations is determinate; namely, only one. But this is not the case with indeterminate equations. Let us take, for example,
$$ax-b+cy=q-x-y,$$

in which x and y are both unknown quantities; hence, the equation may be resolved either with regard to x or with regard to y. Let us resolve it with regard to x; we will have

$$x = \frac{b+q}{a-1} - \frac{c+1}{a+1}y.$$

Now x depends on the value of y, and giving, for example, to y a numerical value equal to 1, the corresponding value of x is

$$x = \frac{b+q}{a-1} - \frac{c+1}{a+1};$$

and giving to y a numerical value equal to 6, the corresponding x is

$$x = \frac{b+q}{a-1} - \frac{c+1}{a-1}6.$$

Unless, therefore, the value of y be determined by some condition, the value of x also remains undetermined, depending on any arbitrary value given to y.

But if two equations are given, containing each the same two unknown quantities x and y, then the value of both can be determined; nay, more generally, when a number of different equations is given equal to the number of the unknown quantities contained in them, all the values of the unknown quantities can be determined.

Equations containing several unknown quantities.

§ 88. The determination of these unknown quantities can be obtained in different manners, as we may see in the following example:

Let each of the equations—

$$\left.\begin{aligned} az + a'y + a''x + a''' &= 0 \\ bz + b'y + b''x + b''' &= 0 \\ cz + c'y + c''x + c''' &= 0 \end{aligned}\right\} (o),$$

contain the same unknown quantities x, y, z; and observe, that any equation of the first degree containing three unknown quantities, can be reduced to the form of the equations (o); so that the different methods applicable to obtain the values of the unknown quantities contained in (o), are applicable to all similar or equivalent cases, and generally to any number of equations containing an equal number of unknown quantities.

Different methods of resolutions.

Resolve, first, each equation with regard to the same unknown quantity; for example, z, we will have

First method. Elimination by comparison.

$$\left.\begin{aligned} z &= -\frac{a'}{a}y - \frac{a''}{a}x - \frac{a'''}{a} \\ z &= -\frac{b'}{b}y - \frac{b''}{b}x - \frac{b'''}{b} \\ z &= -\frac{c'}{c}y - \frac{c''}{c}x - \frac{c'''}{c} \end{aligned}\right\} (o').$$

In this manner, since the first member of each equation (o′) is the same, and, consequently, the second members are all equal to each other, we have

$$-\frac{a'}{a}y - \frac{a''}{a}x - \frac{a'''}{a} = -\frac{b'}{b}y - \frac{b''}{b}x - \frac{b'''}{b},$$

$$-\frac{a'}{a}y - \frac{a''}{a}x - \frac{a'''}{a} = -\frac{c'}{c}y - \frac{c''}{c}x - \frac{c'''}{c}.$$

And, consequently, by transposing the terms to the second member, we have

$$0 = \left(\frac{a'}{a} - \frac{b'}{b}\right)y + \left(\frac{a''}{a} - \frac{b''}{b}\right)x + \frac{a'''}{a} - \frac{b'''}{b},$$

$$0 = \left(\frac{a'}{a} - \frac{c'}{c}\right)y + \left(\frac{a''}{a} - \frac{c''}{c}\right)x + \frac{a'''}{a} - \frac{c'''}{c}.$$

And making

$$\left(\frac{a'}{a} - \frac{b'}{b}\right) = d,\ \left(\frac{a''}{a} - \frac{b''}{b}\right) = d',\ \left(\frac{a'''}{a} - \frac{b'''}{b}\right) = d'',$$

$$\left(\frac{a'}{a} - \frac{c'}{c}\right) = \delta,\ \left(\frac{a''}{a} - \frac{c''}{c}\right) - \delta',\ \left(\frac{a'''}{a} - \frac{c'''}{c}\right) = \delta''.$$

$$\left.\begin{aligned} dy + d'x + d'' &= 0 \\ \delta y + \delta' x + \delta'' &= 0 \end{aligned}\right\} (o'').$$

Now resolving each equation (o″) with regard to y, we have

$$y = -\frac{d'}{d}x - \frac{d''}{d},$$

$$y = -\frac{\delta'}{\delta}x - \frac{\delta''}{\delta};$$

and consequently,

$$-\frac{d'}{d}x - \frac{d''}{d} = -\frac{\delta'}{\delta}x - \frac{\delta''}{\delta},$$

from which $\left(\frac{d'}{d} - \frac{\delta'}{\delta}\right)x + \left(\frac{d''}{d} - \frac{\delta''}{\delta}\right) = 0,$

and making

$$\left(\frac{d'}{d} - \frac{\delta'}{\delta}\right) = \mathrm{D}, \left(\frac{d''}{d} - \frac{\delta''}{\delta}\right) = \mathrm{D}';$$

$$\mathrm{D}x + \mathrm{D}' = 0 \left.\right\} (o'''),$$

which resolved, gives a determinate value of x; namely,

$$x = -\frac{\mathrm{D}'}{\mathrm{D}}.$$

Now this value substituted in one of the preceding (o''), gives us an equation containing only the unknown quantity y, whose value, therefore, can be obtained; and this, together with x substituted in any of the given equations (o), gives us an equation containing the unknown quantity z alone. This method of elimination is called elimination by comparison. Let us pass to the second method.

Elimination by substitution.

The second method consists in resolving one of the given equations (o), for instance, with regard to z, and then substituting the found value of z in the other equations. Thus, we obtain two equations containing only the unknown quantities x and y, to which the same method of elimination can be applied, in order to obtain an equation with only one unknown quantity. But let us see the process of the operation. Resolve the first equation (o) with regard to z, we will have

$$z = -\frac{a'}{a}y - \frac{a''}{a}x - \frac{a'''}{a} \left.\right\} (p).$$

This value of z, substituted in the second and third (o), gives us

$$b\left(-\frac{a'}{a}y-\frac{a''}{a}x-\frac{a'''}{a}\right)+b'y+b''x+b'''=0,$$

$$c\left(-\frac{a'}{a}y-\frac{a''}{a}x-\frac{a'''}{a}\right)+c'y+c''x+c'''=0;$$

from which we deduce the two equivalents

$$\left(b'-\frac{ba'}{a}\right)y+\left(b''-\frac{ba''}{a}\right)x+\left(b'''-\frac{ba'''}{a}\right)=0,$$

$$\left(c'-\frac{ca'}{a}\right)y+\left(c''-\frac{ca''}{a}\right)x+\left(c'''-\frac{ca'''}{a}\right)=0,$$

and making

$$\left(b'-\frac{ba'}{a}\right)=d,\ \left(b''-\frac{ba''}{a}\right)=d',\ \left(b'''-\frac{ba'''}{a}\right)=d'',$$

$$\left(c'-\frac{ca'}{a}\right)=\delta,\ \left(c''-\frac{ca''}{a}\right)=\delta',\ \left(c'''-\frac{ca'''}{a}\right)=\delta'',$$

we will have
$$\left.\begin{aligned}dy+d'x+d''&=0\\ \delta y+\delta'x+\delta''&=0\end{aligned}\right\}(p').$$

Resolve now the first of these two equations with regard to y, we will have
$$y=-\frac{d'}{d}x-\frac{d''}{d};$$

and this value, substituted in the second, gives

$$\delta\left(-\frac{d'}{d}x-\frac{d''}{d}\right)+\delta'x+\delta''=0;$$

from which we have

$$\left(\delta-\frac{\delta d'}{d}\right)x+\left(\delta''-\frac{\delta d''}{d}\right)=0;$$

or making
$$\delta'-\frac{\delta d'}{d}=\mathrm{D},\ \delta''-\frac{\delta d''}{d}=\mathrm{D}'$$

$$\mathrm{D}x+\mathrm{D}'=0\ \}\ (p'')$$

from which,
$$x=-\frac{\mathrm{D}'}{\mathrm{D}}.$$

This value of x substituted in one of the preceding (p), gives us an equation with the unknown quantity y alone, and substituting x and y in (p), we obtain the value of the third unknown quantity z.

Elimination by addition and subtraction.

The elimination by addition and subtraction in some cases is preferable to the two preceding. This method of elimination consists in giving the same coefficient to the same unknown quantity in different equations, and then subtracting one equation from another if the terms affected with the same coefficient have the same sign, or adding the equations if the terms affected with the same coefficient have different signs. Let us resume the equations (o); and first to reduce the unknown quantity z to the same coefficient in the first and second equation, multiply all the terms of the first (o) by the coefficient of z of the second, and all the terms of the second (o) by the coefficient of z of the first, we will obtain the following equations:

$$baz + ba'y + ba''x + ba''' = 0,$$
$$baz + ab'y + ab''x + ab''' = 0,$$

which, subtracted from one another, give

$$y(ba' - ab') + x(ba'' - ab'') + (ba''' - ab''') = 0.$$

In equal manner, reducing to the same coefficient the first term of the second and third (o), we will have

$$cbz + cb'y + cb''x + cb''' = 0,$$
$$cbz + bc'y + bc''x + bc''' = 0,$$

and, consequently,

$$y(cb' - bc') + x(cb'' - bc'') + (cb''' - bc''') = 0,$$

making now

$$(ba' - ab') = d,\ (ba'' - ab'') = d',\ (ba''' - ab''') = d'',$$
$$(cb' - bc') = \delta,\ (cb'' - bc'') = \delta',\ (cb''' - bc''') = \delta''.$$

The obtained equations will become

$$\left.\begin{array}{l} dy + d'x + d'' = 0 \\ \delta y + \delta' x + \delta'' = 0 \end{array}\right\} (q').$$

Reduce now the first term of both (q') to the same coefficient,

we will have
$$d\delta y + d'\delta x + d''\delta = 0,$$
$$d\delta y + d\delta' x + d\delta' = 0,$$
which being subtracted from one another, give
$$x(d'\delta - d\delta') + (d''\delta - d\delta'') = 0,$$
and making $d'\delta - d\delta' = \mathrm{D}$, $d''\delta - d\delta'' = \mathrm{D}'$,
the same equation becomes
$$\mathrm{D}x + \mathrm{D}' = 0 \ \} \ (q'),$$
from which
$$x = -\frac{\mathrm{D}'}{\mathrm{D}},$$
a value which substituted in either of the preceding equations (q'), enables us to find out the value of y; and substituting both x and y in any of the given (o), we obtain an equation with the unknown quantity z alone, which is, consequently, easily determined.

The same methods applicable to all cases.

§ 89. The methods of elimination just described are applicable to any number of equations containing an equal number of unknown quantities. But if the number of the equations is greater than that of the unknown quantities, the resolution may be then impossible; and such equations are then called *incompatible*. Such, for example, are the equations

Incompatible equations.

$$2x + 3y + 4 = 0,$$
$$4x - \ \ y - 6 = 0,$$
$$5x + \ \ y + 2 = 0,$$
which no values of x and y can resolve. Because, adding together the two last equations, we find
$$9x = 4;$$
that is
$$x = \frac{4}{9};$$
and consequently, from the same two last equations,
$$y = -\frac{38}{9}.$$
Now, these two values of x and y to fulfil the first equation,

must make $2x + 3y = -4$; but, making the substitution, we find

$$2x + 3y = \frac{8}{9} - \frac{114}{9} = -\frac{106}{9};$$

hence the three given equations are incompatible; but if the first equation should be

$$9x - 18y - 80 = 0,$$

then the equation would be fulfilled by $x = \frac{4}{9}$ and $y = -\frac{38}{9}$, but this equation is then superfluous for the determination of the values of x and y. We may generally say, therefore, that the number of equations must not be greater than that of the unknown quantities, nor less than the number of the same quantities; although in some instances, with a number of equations less than the number of unknown quantities, we may be able to determine their values. For example, from the equations

$$\begin{aligned} 4x - y - 2z &= 6, \\ 6x + 4y + 8z &= 20, \end{aligned}$$

we may determine the values of the three unknown quantities x, y, z, as follows: Multiply the first equation by 4, and then add together the two equations, we will have

$$22x = 44;$$

and consequently, $x = 2$.

Now the value of x being substituted in both the given equations, we obtain two equations and two unknown quantities, which, consequently, can be determined with any of the preceding methods. We may observe here, also, that many expedients suggested by practice render the resolution of equations containing various unknown quantities in several cases more or less speedy. The general rules, however, given for practice, and deduced from the foregoing processes, are—

Rules and examples.

For the method of elimination by substitution, *Find the value of one unknown quantity in any of the given equations, and substitute it in the others.*

For the method of elimination by comparison:

Find the value of the same unknown quantity in each of the given equations, and form equations with these values.

For the method of elimination by addition and subtraction:

Give the same coefficient to the same unknown quantity in all the equations, and add or subtract as the case may require.

Examples, or given equations:

$$(1.)\quad \begin{cases} 2x + 3y + 4z = 29, \\ 3x + 4y + 5z = 38, \\ 5x - 2y + 2z = 12. \end{cases}$$

$$(2.)\quad \begin{cases} \frac{9}{10}x - \frac{2}{3}y - 16 = 0, \\ \frac{1}{10}x + 4y - 14 = 0. \end{cases}$$

$$(3.)\quad \begin{cases} 2x + 3y + 4z = 16, \\ 3x + 2y - 5z = 8, \\ 5x - 6y + 3z = 6. \end{cases}$$

$$(4.)\quad \begin{cases} 5x - 6y + 4z = 15, \\ 7x + 4y - 3z = 19, \\ 2x + y + 6z = 46. \end{cases}$$

Answers:

(1.) $x = 2,\ y = 3,\ z = 4.$
(2.) $x = 20,\ y = 3.$
(3.) $x = 3,\ y = 2,\ z = 1.$
(4.) $x = 3,\ y = 4,\ z = 6.$

Sometimes, not all the unknown quantities are to be found in each of the given equations; as, for instance, in the annexed example:

$$\begin{aligned} ax + a'y &= a'' \\ bx + b'z &= b'' \\ cy + c'z &= c''. \end{aligned}$$

But from the first and second we can eliminate the unknown quantity x, and have an equation containing y and z, which, together with the third, will give us the values of the same unknown quantities. In a similar manner we will find the values of the unknown quantities contained in the following examples:

Given equations:

$$(5.)\quad \left\{\begin{array}{l} 2x - 4y + 8z = 54, \\ 12y - 7z + 8 = 0, \\ 4x - 36 = - 3z. \end{array}\right.$$

$$(6.)\quad \left\{\begin{array}{r} 2x - 3y + 2z = 13, \\ 4v - 2x = 30. \\ 4y + 2z = 14, \\ 5y + 3v = 32. \end{array}\right.$$

$$(7.)\quad \left\{\begin{array}{l} x + y = 36, \\ x + z = 49, \\ y + z = 53. \end{array}\right.$$

Answers:

(5.) $x = 3,\ y = 4,\ z = 8.$

(6.) $x = 3,\ y = 1,\ z = 5,\ v = 9.$

(7.) $x = 16,\ y = 20,\ z = 33.$

Problems.

§ 91. Problems frequently contain more than one unknown quantity. In this case the conditions of the problem must commonly contain as many equations as there are unknown quantities to be determined. The skill required in the resolution of the problem consists in knowing how to give the algebraical form to the equations problematically expressed.

Practice and natural aptitude, rather than any rule, facilitate the resolution of problems. We may, however, observe that the difficulty in the resolution of problems is greatly diminished by this general rule:

General rule.

Separate first the unknown quantity, and then modify and combine them according to the conditions of the problem.

An application of this rule may be seen in the following ex[illegible]ple:

Problem 1. The three ciphers of a number are such that their sum is 14; the sum of the first and last divided by the second, gives 6; and subtracting 594 from the given number, the difference contains the same three unknown ciphers, disposed in an inverted order. What is the number?

The three figures of the number are the unknown quantities of the problem, which we separate from the known quantities contained in the problem, calling them x, y, z. Now the first condition is, that the sum of the figures is equal to 14. Hence, the first equation

$$x + y + z = 14 \quad (1).$$

Another condition expressed in the problem is, that the sum of the first and third figures, divided by the second, gives 6 for quotient; hence, the second equation $\frac{x+z}{y} = 6$, or

$$x + z = 6y \quad (2).$$

The last condition is, that subtracting 594 from the unknown numbers, the remainder is the same unknown number taken in an inverted order. The equation contained in this condition is not so obvious as the preceding; to deduce it, observe that the number 594 may be decomposed, as follows:

$$\begin{aligned} 594 &= 500 + 90 + 4 \\ &= 100.5 + 10.9 + 4; \end{aligned}$$

hence, the number also, whose first cipher is x, the second y, and the last z, is likewise resolvable into three; namely,

$$100.x + 10.y + z;$$

and therefore, the inverted number is

$$100.z + 10.y + x.$$

Hence, the equation contained in the last condition is

$$100x + 10.y + z - 594 = 100.z + 10.y + x,$$

or

$$99x - 99z = 594$$

which is easily reduced to

$$x - z = 6 \quad (3).$$

Thus we have obtained as many equations as there are unknown quantities. To have them resolved, subtract first equation (2) from equation (1); we will have

$$y = 14 - 6y;$$

that is, $$y = 2;$$

which value of y, substituted in equation (2), gives

$$+ z = 12.$$

Subtract now from this, and then add to the same equation (3), we will have

$$2z = 6,\ 2x = 18;$$

that is, $$z = 3,\ x = 9;$$

the required number, therefore, is

$$N = 923.$$

We have, in fact,

$$x + y + z = 9 + 2 + 3 = 14$$

$$\frac{x + z}{y} = \frac{9 + 3}{2} = 6$$

$$923 - 594 = 329.$$

Problem 2. What two numbers are those whose difference is 9, and sum three times as much?

Ans. $x = 18$, $y = 9$.

Problem 3. What three numbers x, y, z are those whose sum is 34; the sum of the last, and twice the first is 30; and the sum of the first and twice the second is 26?

Ans. $x = 6$, $y = 10$, $z = 18$.

Problem 4. The weight of four globes A, B, C, D is 340 pounds, and the weight of A + D is equal to that of B + C; C is ten pounds less than B; and the weight of D, plus one-third that of B, make the weight of A. What is the weight of each globe?

Ans Calling x, y, v, z the respective weights of A, B, C, D, we will find $x = 100$, $y = 90$, $v = 80$, $z = 70$.

Problem 5. In a mixture of wine and water, one-tenth of the whole, plus 10 gallons, is water, and one-half of the whole, plus 30 gallons, is wine. How many gallons are there of each? Ans. $x = 80$, $y = 20$.

Problem 6. Divide the number 144 into four such parts, that, if the first be divided by 5, and the second multiplied by 5, the third diminished by 5, and the last increased by 5, the quotient, the product, the difference, and the sum are all equal.

Ans. Calling x, y, v, z the first, second, third, and fourth parts, $x = 100$, $y = 4$, $v = 25$, $z = 15$.

Problem 7. Three persons A, B, C have each a certain sum of money: one-third of the money of A and C, minus 6 dollars, is the sum of B; one-half the money of C, minus the money of A, and minus 9 dollars, give, again, the sum of dollars of B; the sum of C, multiplied by $\frac{2}{9}$, gives twice the sum of A. What is the sum of each?

Ans. Calling x, y, z the sums of A, B, C, we have

$x = 18$, $y = 54$, $z = 162$.

Problem 8. Five wheels A, B, C, D, E are so combined, that while A performs x revolutions, B performs y, C performs v, D, w, and E, z. Now ten times the revolutions of A, plus three times those of B, and four times those of E, give the same number as 9 times the revolutions of D, plus the product of the number of revolutions of C multiplied by $\frac{33}{10}$; twice the revolutions of A, plus twice those of C, give the same number as the revolutions of D, added to one-fourth of those of E; the revolutions of D and E, plus ten times those of B, are equal to seven times the revolutions of C; the revolutions of A, plus five times those of B, give the revolutions of C; and the revolutions of E, minus three times those of C, give 20 revolutions. How many revolutions does each wheel perform in the same time?

Ans. $x = 10,\ y = 2,\ v = 20,\ w = 40,\ z = 80$

Problem 9. What fraction is that whose value is $\frac{1}{3}$, if we add 1 to its numerator, and $\frac{1}{4}$, if we add one to its denominator?

Ans. $\frac{x}{y} = \frac{4}{15}$.

Problem 10. There are two horses and two saddles: the best saddle costs 40 dollars, and the other only 6; placing the best saddle on the first horse, and the other on the second, the first horse costs 6 dollars more than the other; and changing the saddles, the second horse costs three times more than the first. What is the price of each horse?

Ans. $x = 25,\ y = 53$.

ARTICLE II.

Equations of the Second Degree.

General formula of the determinate equations of the second degree.

§ 92. Any equation in which the highest exponent of the unknown quantity or quantities is 2, is (84) an equation of the second degree. Hence, the general formula of the equations of the second degree containing only one unknown quantity, is

$$x^2 + Ax = B \quad (h);$$

because all the terms to be possibly found in this class of equations are either known quantities or terms containing the simple unknown quantity x, and terms containing the square of x; as, for instance, in the equation

$$mx^2 - nx + px^2 - q = x + rx^2 - fx + g,$$

in which q and g represent known quantities, like the coefficients

m, n, p, r, f; the terms nx, x, fx contain the simple unknown quantity x; and the remaining terms the square of x. It is now easy to reduce this equation to the form of the preceding (h), because, through a simple transposition, we can first write

$$mx^2 + px^2 - rx^2 - nx - x + fx = q + g;$$

that is, $(m + p - r)x^2 + (f - n - 1)x = q + g$,

from which $$x^2 + \frac{f - n - 1}{m + p - r}x = \frac{q + g}{m + p - r},$$

an equation of the same form as (h). Hence, the resolution of any equation of the second degree is the same as the resolution of the general equation (h); and the first operation to be made when an equation of the second degree, containing only one unknown quantity, is given to be resolved, is to reduce the given equation to the form of (h); hence, also, the first and general rule:

General rule. *Transpose all the known terms to the second member, and all the others to the first; reduce to a single term all those that contain the square of the unknown quantity, and likewise all those which contain the first power of the same quantity; then clear the square of the unknown quantity of its coefficient.*

Remark. It is to be observed that equations of the second degree do not sometimes apparently contain terms with the square of the unknown quantity; as, for instance, in the equation

$$ax - b = \frac{n}{x};$$

but the same equation is reducible to the following:

$$ax^2 - bx = n.$$

In cases similar to this, before proceding to resolve the equation, a similar transformation is to be made.

The given equations being thus prepared, we may pass to see how the general equation (h) is resolved.

Resolution of the general equation. Two cases.

§ 93. Two cases can take place with regard to the first member of (h), according as the coefficient A is either equal or not to zero. In the first case the equation is simplified, and becomes

First case.

$$x^2 = B \quad (h'),$$

which is easily resolved, because, from (h'), we have $\sqrt{x^2} = \pm\sqrt{B}$, or $\qquad x = \pm\sqrt{B}$.

That is, the value of the unknown quantity x is the positive as well as the negative square root of the known quantity B, or the second member of (h'). We have, in fact,

$$[+\sqrt{B}]^2 = +B,$$
$$[-\sqrt{B}]^2 = +B;$$

both values, therefore, $+\sqrt{B}$, $-\sqrt{B}$, fulfil the equation (h'). When, therefore, the given equation, reduced to the general form, assumes that of (h'), the double value of the unknown quantity is immediately found, as follows:

Rule and examples.

Take the positive and negative square root of the second member.

The equations, for example,

(1.) $3x^2 - 3 = 6 - 2x^2$,
(2.) $4x^2 - 36 = 2x^2 - 4$,
(3.) $2ax^2 - x + b = ax^2 - x + 3b$,

reduced to the general formula, give

(1.) $x^2 = 9$,
2.) $x^2 = 16$,
(3.) $x^2 = \frac{2b}{a}$,

and resolved,

(1.) $x = \pm\sqrt{9} = \pm 3$.
(2.) $x = \pm\sqrt{16} = \pm 4$.
(3.) $x = \pm\sqrt{\frac{2b}{a}}$.

Second case. But when A is not equal to zero, the equation (*h*) cannot generally be resolved without being modified, as we will presently see.

Observe that (69) $(x+y)^2 = x^2 + 2xy + y^2$, and the second term of this evolution is the double product of the two terms of the binomial; so that the last term of the same evolution can be easily inferred from the first and the second, by dividing, namely, this by the double root of the first, and squaring the quotient. For example, $a^2 + 2ma$, representing the incomplete evolution of the square of a binomial, it will become complete by adding m^2 to it, because $\frac{2ma}{2a}$ gives m for quotient; and consequently, m^2 for the last term of the unfinished evolution. In like manner, $b^2 + cb$, representing another unfinished evolution, by adding to it the term $\frac{c^2}{4}$, it will become complete. That is, to render $b^2 + cb$ a perfect square, add the square of half the coefficient of b.

Now, the first member of the general formula (*h*), or $x^2 + \mathrm{A}x$, has the form of an incomplete square, which is easily finished by adding $\frac{\mathrm{A}^2}{4}$, the square of half the coefficient of x. But in order to preserve the equality, if we finish the square in the first member, we must add the same term to the second member also. This addition being made, we will have

$$x^2 + \mathrm{A}x + \frac{\mathrm{A}^2}{4} = \mathrm{B} + \frac{\mathrm{A}^2}{4} \quad (h'');$$

or, since

$$x^2 + \mathrm{A}x + \frac{\mathrm{A}^2}{4} = \left(x + \frac{\mathrm{A}}{2}\right)^2;$$

$$\left(x + \frac{\mathrm{A}}{2}\right)^2 = \mathrm{B} + \frac{\mathrm{A}^2}{4};$$

from which

$$x + \frac{\mathrm{A}}{2} = \pm\sqrt{\mathrm{B} + \frac{\mathrm{A}^2}{4}};$$

and consequently,

$$x = -\frac{A}{2} \pm \sqrt{B + \frac{A^2}{4}} \quad (h'''),$$

the values of x, which resolve the equation (h) given by the known quantities A and B. We say the values, because, when we take the upper sign,

$$x = -\frac{A}{2} + \sqrt{B + \frac{A^2}{4}};$$

when we take the other,

$$x = = \frac{A}{2} + \sqrt{B + \frac{A^2}{4}}$$

Real and imaginary roots.

These values of x, or roots of the equation (h) will be either both real or both imaginary; and again, when real, both positive or negative, or one positive and one negative.

Let us here examine all these cases. When the binomial $B + \frac{A^2}{4}$, under the radical sign, is a positive quantity, the radical, and consequently the value of x in both equations, is real. But when the same binomial is negative, then (50) the radical is imaginary, and the values of x contain the real term $-\frac{A}{2}$, plus or minus the imaginary root; and, therefore, both values of x are imaginary, because neither a positive nor a negative real term or quantity can ever represent the difference or the sum of two expressions, the one real, and the other imaginary.

Suppose, first, the binomial $B + \frac{A^2}{4}$ to be a positive quantity; in this supposition B is positive, or if negative, less than $\frac{A^2}{4}$. When B is positive, one of the values of x is positive and the other negative; when B is negative, the values of x are either both positive or both negative. We have, in fact, in the first case,

$$\sqrt{B + \frac{A^2}{4}} > \sqrt{\frac{A^2}{4}} \left(= \frac{A}{2}\right);$$

and, therefore, we may write

$$\pm\sqrt{B+\frac{A^2}{4}}=\pm\left(\frac{A}{2}+d\right);$$

hence, from (h'''), if the coefficient A in (h) is positive,

$$x=-\frac{A}{2}+\frac{A}{2}+d=d,$$

$$x=-\frac{A}{2}-\frac{A}{2}-d=-A-d;$$

and if the coefficient A in (h) is negative,

$$x=+\frac{A}{2}+\frac{A}{2}+d=A+d,$$

$$x=+\frac{A}{2}-\frac{A}{2}-d=\quad -d.$$

First criterion. Hence, *When in the equation* (h), *that is,*

$$x^2+Ax=B;$$

B *is positive, one of the roots is likewise positive, and the other negative, whatever be the sign of the coefficient* A.

In the second case, when B is negative and less than $\frac{A^2}{4}$, we have

$$\sqrt{B+\frac{A^2}{4}}<\sqrt{\frac{A^2}{4}}\left(=\frac{A}{2}\right);$$

that is,

$$\sqrt{B+\frac{A^2}{4}}=\frac{A}{2}-d;$$

therefore, if the coefficient of x in (h) is positive, we will have from (h''')

$$x=-\frac{A}{2}+\frac{A}{2}-d=-d,$$

$$x=-\frac{A}{2}-\sqrt{B+\frac{A^2}{4}},$$

both values of x being negative.

In the same supposition of B negative, if the coefficient A of x also is negative, we will have from (h''')

$$x=\frac{A}{2}+\sqrt{B+\frac{A^2}{4}},$$

$$x=\frac{A}{2}-\frac{A}{2}+d=+d,$$

both values of x being positive. Therefore,

Second criterion. *When in the equation* (h), B *is negative and less than* $\frac{A^2}{4}$, *and* A *is positive, the roots are both real and negative. With the same* B *and with* A *negative, the roots are both real and positive.*

Let us now suppose B negative, and equal to $\frac{A^2}{4}$. In this case,

$$\sqrt{B+\frac{A^2}{4}}=0;$$

and consequently, both values of x from (h''') are equal to each other, and have the same sign. That is,

Third criterion. *When in* (h), B *is negative and equal to* $\frac{A^2}{4}$, *both values of the roots are equal to* $\frac{A}{2}$ *positive, being the coefficient of* x *negative, or equal to* $\frac{A}{2}$ *negative when the coefficient of* x *is positive.*

The last case is that of B negative, and greater than $\frac{A^2}{4}$, in which case the binomial $B+\frac{A^2}{4}$ is necessarily a negative quantity, $-d$; hence,

$$\sqrt{B+\frac{A^2}{4}}=\sqrt{-d},$$

an imaginary expression. Therefore,

Fourth criterion. *When in* (h) *the value of* B *is negative, and greater than* $\frac{A^2}{4}$, *the roots of the equation are both imaginary.*

The preceding criterions applied to the equations—

(1.) $x^2+7x=12,$
(2.) $x^2-15x=18,$
(3.) $x^2+16x=-40,$
(4.) $x^2-20x=-90,$
(5.) $x^2+12x=-36,$
(6.) $x^2-8x=-16,$
(7.) $x^2+14x=-50,$
(8.) $x^2-6x=-18,$

reduced already to the general form (h), show that the roots of the equations (1) and (2) are real, and affected with different signs; the roots of (3) are also real, and both negative; the roots of (4) are real, and both positive; those of (5) are also both negative, and, besides, equal to each other; those of (6) are both positive, and equal to each other. The roots of the equations (7) and (8) are imaginary.

Examples and problems. § 94. Let us proceed to resolve some equations. Given equations:

(1.) $2x^2+x=x^2+54-2x.$
(2.) $x^2-x=56.$

Given equations,

(3.) $2x^2 - 4x - 9 = x^2 + 2x - 17.$

(4.) $16x^2 - 4x + 36 = 14x^2 - 32x - 60.$

(5.) $x^2 - ab + ax = bx.$

(6.) $x^2 + mn - nx = mx.$

(7.) $x^2 + cx - c^2 = -2cx - 3c^2.$

(8.) $x^2 + 26 = 4x + 13.$

(9.) $x^2 - 2ax + a^2 + b^2 = 0.$

The first of the proposed examples is easily reduced to the general form (h), as follows:

$$x^2 + 3x = 54,$$

and adding the term $\left(\frac{3}{2}\right)^2$ in order to have a complete square in the first member, we have

$$x^2 + 3x + \frac{9}{4} = 54 + \frac{9}{4},$$

or
$$\left(x + \frac{3}{2}\right)^2 = 54 + \frac{9}{4};$$

and consequently,

$$x + \frac{3}{2} = \pm\sqrt{\frac{225}{4}} = \pm\frac{15}{2};$$

hence, the double value of x:

$$1.) \left\{ \begin{array}{l} x = -\frac{3}{2} + \frac{15}{2} = 6, \\ x = -\frac{3}{2} - \frac{15}{2} = -9. \end{array} \right.$$

The process is the same for the other examples, and is contained in the annexed practical rule:

Rule. *Reduce the given equation to the general form, finish the square of the first member, extract the root of both members, and leave the unknown quantity alone in the first member.*

In this manner, the remaining examples, being resolved, will give:

$$(2.)\begin{cases} x = \\ x = -8. \end{cases} \qquad (3.)\begin{cases} x = 2 \\ x = 4. \end{cases}$$

$$(4.)\begin{cases} x = -6 \\ x = -8. \end{cases} \qquad (5.)\begin{cases} x = -a \\ x = +b. \end{cases}$$

$$(6.)\begin{cases} x = m \\ x = n \end{cases} \qquad (7.)\begin{cases} x = -c \\ x = -2c. \end{cases}$$

$$(8.)\begin{cases} x = 2 + 3\sqrt{-1} \\ x = 2 - 3\sqrt{-1}. \end{cases} \qquad (9.)\begin{cases} x = a + b\sqrt{-1} \\ x = a - b\sqrt{-1}. \end{cases}$$

Problems. When the conditions of a problem, whose resolution is reducible to that of an equation of the second degree, are such as to exclude, for instance, the negative sign for the unknown quantity, and the equation resolved gives the values of the unknown quantity affected with opposite signs, the positive alone resolves the problem.

Problem 1. The square number of my dollars added to 180, gives 27 times the number of my dollars. How many dollars have I? Ans. $x = 12$, or $x = 15$.

Problem 2. I have as many dogs as he has cats. All my dogs, plus four of his cats, multiplied by the whole number of dogs and cats, give 12 times the number of dogs, plus 160. What is the number of my dogs?

Ans. The equation resolved gives $x = -8$, $x = 10$; the first value is to be excluded. Hence, $x = 10$.

Problem 3. Find a number whose product by 5, minus six units, mutiplied by the same number added to 1, gives for product seven times its negative square.

Ans. $x = +\frac{3}{4}$, $x = -\frac{2}{3}$.

Problem 4. The product of a certain number by 7, minus 75, is equal to 95, plus the quotient arising from 125 divided by the same number. What is the number?

Ans. $x = +25$, $x = -\frac{5}{7}$.

Problem 5. An army commencing battle, contains an equal number of men in each rank, and it contains as many ranks as there are men in one rank. During the battle, the first three ranks and 350 men beside are killed. The army after the battle contains 2000 soldiers. Find the original number.

Ans. $x = 2500$, each rank containing 50 men.

With regard to equations of the second degree, containing more than one unknown quantity, the same methods of elimination given in the preceding number (88) can be applied.

ARTICLE III.

On some Properties of Determined Equations of any Degree.

Preliminary theorems. § 95. Several discussions of the present article rest on some general theorems, useful for other investigations, no less than to find out the properties of equations of any degree. We commence, therefore, this article by demonstrating the same theorems. And, first,

Let the coefficients $a, b, c, \ldots h.$ of the variable and real quantity z, and the last term k of the polynomial

$$az^n + bz^{n-1} + cz^{n-2} + \ldots hz + k \ldots\ldots (p),$$

be all real and invariable quantities. We say, that if by changing the value of z, (p) assumes a positive and then a negative value,

Theorem 1. *There must be some value* z_m *of* z, *which substituted in* (p), *makes this polynomial equal to zero.*

To demonstrate this proposition, let us suppose the value of z to be changed in such a manner that the difference between any two such values, taken in succession, be capable of an indefinite diminution. In this manner the polynomial (p) also will be changed by degrees capable of indefinite attenuation. But by supposition, the polynomial (p) may be changed from positive into negative; so that, making, for instance, $z = z_h$; the polynomial assumes the positive value $(+ p_h)$:

and making $z = z_n$, the polynomial becomes negative, that is, $(-p_n)$. The difference, therefore, between the two values of the polynomial (p), is

$$(p_h) + (p_n),$$

which may become smaller and smaller either by z_h approaching to z_n, or z_n to z_h, or both of them to each other. Suppose that leaving z_n unchanged, z_h approaches constantly to z_n, the difference $(p_h) + (p_n)$ will indefinitely approach to zero, and by degrees capable of indefinite attenuation; that is to say, the said difference is capable of assuming all the values contained between $(p_h) + (p_n)$ and zero; now (p_n) is one of these values; therefore, among the values which the difference $(p_h) + (p_n)$ will take by z_h approaching incessantly to z_n, is also (p_n); and since the difference $(p_h) + (p_n)$ cannot become (p_n) unless (p_h) becomes zero, therefore, the value of (p_h) constantly changed with z_h will once become zero. Call z_m the value which makes $(p_h) = 0$, we will have

$$az_m^n + bz_m^{n-1} + cz_m^{n-2} + \dots + hz_m + k = 0.$$

Theorem 2. *When the decrease of the variable* z *in* (p) *is carried to a certain limit, the polynomial retains from that limit constantly the same sign, equal to the sign of its last term.*

The polynomial (p) without its last term is

$$az^n + bz^{n-1} + cz^{n-2} + \dots + hz,$$

which evidently approaches to zero by constantly diminishing the value of z. Now it cannot uninterruptedly approach to zero without becoming smaller than any fixed value different from zero; hence, by diminishing constantly in (p), the value of z, all its terms, with the exception of the last k, will finally become a smaller quantity than the same k. And, consequently, from this limit, whatever might be the sign of the rest, the sign of the whole polynomial (p) will be that of k, if k is positive; (p) also, from that limit, will be constantly positive; if k is negative, (p) from the same limit will be also negative.

Theorem 3. *When* z *in* (p) *is increased to a certain limit, the polynomial from that limit will constantly retain the sign of its first term.*

The polynomial (p) is manifestly equivalent to the following product:

$$z^n\left(a + \frac{b}{z} + \frac{c}{z^2} + \dots + \frac{h}{z^{n-1}} + \frac{k}{z^n}\right).$$

Now, by constantly increasing the value of z, each term within the parenthesis, except the first, approaches constantly to zero, and, consequently, also the sum of all of them. Hence, the same sum, when z is increased to a certain value, will be equal to and then become smaller than the fixed quantity a. If now, for the sake of brevity, we call S the sum of the diminishing terms, we will have

$$(p) = z^n(a + \mathrm{S}).$$

In which, when z is increased to the said limit, and much beyond that limit, S is smaller than a; hence, from this limit, the sign of $a + \mathrm{S}$ must be the same as that of a, whatever be the sign of S; hence, also, the sign of the product $z^n(a + \mathrm{S})$, that is, of the polynomial (p), is the same as the sign of az^n, which is the first term of the same polynomial.

Theorem 4. *When two polynomials, such as*

$$\left.\begin{array}{l} a_0 + a_1z + a_2z^2 + \ldots + a_nz^n \\ c_0 + c_1z + c_2z^2 + \ldots + c_nz^n \end{array}\right\} (\mathrm{P}),$$

remain equal to each other, substituting in them n + 1 *different values of* z, *the two polynomials are identical.*

Let $z_0, z_1, z_2, \ldots . z_n$ represent the $n + 1$ different values, which, substituted in succession in the polynomials, make them equal to each other; that is,

$$a_0 + a_1z_0 + a_2z_0^2 + \ldots + a_nz_0^n = c_0 + c_1z_0 + c_2z_0^2 + \ldots . + c_nz_0^n,$$

$$a_0 + a_1z_1 + a_2z_1^2 + \ldots + a_nz_1^n = c_0 + c_1z_1 + c_2z_1^2 + \ldots . + c_nz_1^n, \text{ \&c.}$$

Hence, also, calling $p_0, p_1, p_2, \ldots .$ the first members of these equations, we will have

$$\left.\begin{array}{l} a_0 + a_1z_0 + a_2z_0^2 + \ldots + a_nz_0^n = p_0 \\ a_0 + a_1z_1 + a_2z_1^2 + \ldots + a_nz_1^n = p_1 \\ a_0 + a_1z_2 + a_2z_2^2 + \ldots + a_nz_2^n = p_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_0 + a_1z_n + a_2z_n^2 + \ldots + a_nz_n^n = p_n \end{array}\right\} (\mathrm{A}),$$

and likewise,

$$\left.\begin{array}{l} c_0 + c_1z_0 + c_2z_0^2 + \ldots . + c_nz_0^n = p_0 \\ c_0 + c_1z_1 + c_2z_1^2 + \ldots . + c_nz_1^n = p_1 \\ c_0 + c_1z_2 + c_2z_2^2 + \ldots . + c_nz_1^n = p_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ c_0 + c_1z_n + c_2z^{n2} + \ldots . + c_nz_n^n = p_n \end{array}\right\} (\mathrm{C})^2.$$

Now the values of $a_0, a_1, a_2, \ldots a_n$, inferred from the $n + 1$ equations (A), are evidently the same as the values of $c_0, c_1, c_2, \ldots c_n$, inferred from the $n + 1$ equations (C). Therefore, from the supposed equality of the polynomials (P), when substituting in them $n+1$ different values of z, it follows, that

$$a_0 = c_0,\ a_1 = c_1,\ a_2 = c_2,\ \dots\ a_n = c_n \ \{\ (h).$$

Now this equality makes the polynomials (P) identical.

Corollary. Hence, when the polynomials (P) are found to be equal to one another in whatsoever manner the variable z be taken, we must necessarily infer the equations (h).

General formula of a determined equation of any degree. § 96. Let the coefficients A, B, C, ... H, and the last terms K of the equation

$$[e]\ x^n + Ax^{n-1} + Bx^{n-2} + \dots + Hx + K = 0,$$

of the n^{th} degree be real quantities. If taking for x two real values, the one makes [e] positive, and the other negative, the same [e] is resolvable with at least one real root; that is, there is at least one real value of x, which makes the first member of [e] equal to zero. To prove this, it is enough to apply to [e] the demonstration of the first theorem of the preceding number.

Equations resolvable with at least one real root. Let us now see how one of two values of x makes [e] positive, and the other negative. When the degree n of the equation is an uneven number, the sign of the first term x^n is the same as the sign of x; but when x has a sufficiently great value according to the third theorem of the preceding number, the sign of the whole polynomial [e] is the same as that of the first term; hence, positive if x is positive, negative if x is negative. When, therefore, the degree of the equation is uneven, the first member of [e], by the substitution of one certain value of x, can be made positive, and by another negative; hence, the equation in this case is certainly resolvable with at least one real root.

Equations resolvable with at least two real roots. When the degree n of the equation is an even number, and the last term k is negative, the equation is then resolvable with at least two real roots. Because, taking a very small value for x, the sign of the formula [e], according to the second among the preceding theorems, is the same as that of its last term K; that is, negative. And taking a sufficiently great value of x, then the sign of the polynomial is the same as that of the first term. But the sign of the first term is positive, whether the value of x be positive or negative: therefore, a value of x between one very small and another large and positive, will make the first member of [e] equal to zero; and again, another value of x between the same very small value and another large and negative, will make the first member of [e]

equal to zero. When, therefore, the degree of the equation [e] is an even number, and its last term K a negative quantity, the equation [e] $= 0$ can be resolved with at least two real roots.

When equations cannot be resolved with real roots, they are resolvable with one or more imaginary roots.

§ 97. We have seen that when the degree of the equation [e] $= 0$ is an uneven number, the equation is always resolvable with at least one real root; and when the degree n of the same equation is an even number, the resolution may be obtained with two different real roots, provided the last term K of [e] be a negative quantity; but if K should be positive, and n an even number, we would then be unable to demonstrate the possibility of resolution of [e] $= 0$ with real roots. Because, although [e] involves a function of x, and always of the same real form, and consequently in the equation [e] $= 0$ the variable x is necessarily a reciprocal function of A, B, C, ... K, that is,

$$x = f(A, B, \dots H, K),$$

of a determined and unvariable form, it may occur that the change of the sign of k changes the real value of x into an imaginary one. For example, the real value of the expression

$$x = \sqrt{A + B},$$

in which A and B are supposed to be positive, becomes imaginary when B, being greater than A, has its sign changed.

But whatever be the value of x, either real or imaginary, it is certain that by substituting in [e] $= 0$ instead of x the function $f(A, B, .. H, K)$ the equation is fulfilled; and, therefore, when the degree of the equation [e] $= 0$ is an even number, and the last term K is positive, the equation is resolvable, at least with an imaginary root.

Last supposition.

The last supposition which can be made with regard to the last term K of the general equation [e] $= 0$ is, that the same term be equal to zero. In this supposition our equation becomes equivalent to the following one:

$$x[Ax^{n-1} + Bx^{n-2} + \dots + H] = 0,$$

or (making $Ax^{n-1} + Bx^{n-2} + \dots + H = [e']$) equivalent to

$$x[e'] = 0.$$

It is now plain that any value of x which makes (e'] $= 0$, makes also $x[e'] = 0$. That is, any root which fulfils the equation [e'] $= 0$, fulfils also the equation [e] $= 0$. But [e'] is a polynomial, having the same form as $[e]$; and we have seen above, that the equation $[e]=0$ admits always at least of one root, either real or imaginary. Therefore, the same equation is likewise resolvable when $K=0$.

Any determined equation of any degree admits of as many roots as there are units in the degree of the equation.

§ 98. We may now pass to see that the number of roots of any determined equation is always equal to the degree of the same equation.

We have seen that the equation $[e]=0$ admits in all cases of at least one root; call h this root, and from $[e]=0$, we will have

$$h^n+Ah^{n-1}+Bh^{n-2}+\ldots\ldots+Hh+K=0;$$

and consequently,

$$K=-h^n-Ah^{n-1}-Bh^{n-2}-\ldots\ldots-Hh.$$

Substituting now this value of K in $[e]$, we will have

$$[e]=x^n+Ax^{n-1}+Bx^{n-2}+\ldots\ldots+Hx-h^n-Ah^{n-1}-Bh^{n-2} \\ -\ldots-Hh,$$

or,

$$[e]=(x^n-h^n)+A(x^{n-1}-h^{n-1})+B(x^{n-2}-h^{n-2})+\ldots+H(x-h),$$

in which x may have any value.

Now we have seen (63), among the last examples of multiplication, that $(1+z+z^2+\ldots+z^n)(1-z)=1-z^{n+1}$, from which, by changing the signs of both members, and taking $n-1$ instead of n, we infer $z^n-1=(1+z+z^2+\ldots+z^{n-1})(z-1)$;

or, substituting $\frac{x}{h}$ instead of z,

$$\left(\frac{x}{h}\right)^n-1=\left(1+\frac{x}{h}+\left(\frac{x}{h}\right)^2+\ldots\ldots+\left(\frac{x}{h}\right)^{n-1}\right)\left(\frac{x}{h}-1\right),$$

and from this

$$\frac{x^n-h^n}{h^n}=\left(\frac{x-h}{h}\right)\left(1+\frac{x}{h}+\left(\frac{x}{h}\right)^2+\ldots+\left(\frac{x}{h}\right)^{n-1}\right);$$

or $\quad x^n-h^n=(x-h)(h^{n-1}+h^{n-2}x+h^{n-3}x^2+\ldots+x^{n-1})$.

Inverting the order of the terms of the last polynomials, and then substituting in succession $n-1$, $n-2$, $n-3$, ... instead of n, we will have

$$x^n-h^n=(x-h)(x^{n-1}+hx^{n-2}+h^2x^{n-3}+\ldots+h^{n-1})$$
$$x^{n-1}-h^{n-1}=(x-h)(x^{n-2}+hx^{n-3}+h^2x^{n-4}+\ldots+h^{n-2})$$
$$x^{n-2}-h^{n-2}=(x-h)(x^{n-3}+hx^{n-4}+h^2x^{n-5}+\ldots+h^{n-3}),\ \&c.$$

Making now, a substitution of the values of these binomials in the expression of $[e]$ last obtained, we will have

$$[e] = (x-h)(x^{n-1}+hx^{n-2}+\ldots.)+A(x-h)(x^{n-2}+hx^{n-3}+\ldots) \\ + B(x-h)(x^{n-3}\ldots.)+\ldots+(x-h)H;$$

or else,

$$[e] = (x-h)[x^{n-1}+(h+A)x^{n-2}+(h^2+Ah+B)x^{n-3}+\ldots \\ +(h^{n-1}+Ah^{n-2}+Bh^{n-3}+\ldots.H)];$$

or more simply,

$$[e] = (x-h)[x^{n-1}+B_1x^{n-2}+C_1x^{n-3}+\ldots+H_1x+K_1]:$$

making, namely, $h+A=B_1$, $h^2+Ah+B=C_1$, &c.

Represent now the last polynomial by $[e_1]$, that is, make

$$x^{n-1}+B_1x^{n-2}+C_1x^{n-3}+\ldots+H_1x+K_1=[e_1];$$

then we will have $\qquad [e]=(x-h)\,[e_1]$,

whatever be the value of x.

Therefore, h being a root of the equation $[e]=0$, the polynomial can be decomposed in two factors, one of which is $x-h$, the second another polynomial $[e_1]$ of the same form as $[e]$, but of a degree one unity lower than that of $[e]$.

Now, resuming again $[e]=0$, or its equivalent

$$(x-h)\,[e_1]=0,$$

it is plain, that not only by making x equal to h, we will have the equation fulfilled; but also, any value of x which renders $[e_1]$ equal to zero, fulfils likewise the equation; that is to say, any root of the equation, $\qquad [e_1]=0$,

is a root also of $[e]=0$. Now, $[e_1]=0$ admits certainly of at least one either real or imaginary root, which we may call i; and applying to $[e_1]=0$ that which we have said with regard to $[e]=0$, we will have $\quad [e_1]=(x-i)[x^{n-2}+C_2x^{n-3}+\ldots.+H_2x+K_2]$,

and making $\qquad x^{m-2}+C_2x^{n-3}+\ldots+K_2=[e_2]$

$$[e_1]=(x-i)[e_2];$$

and since $[e]=(x-h)[e_1]$, also

$$[e]=(x-h)(x-i)\,[e_2].$$

But the polynomial $[e_2]$, like the other two $[e]$ and $[e_1]$, can be decomposed into two factors, the first of which having the form $(x-l)$, and the second the same form as the preceding polynomials $[e]$, $[e_1]$, $[e_2]$, with this difference, that the highest degree of x in $[e]$ is n; in $[e_1]$, $n-1$; in $[e_2]$, $n-2$, in the following polynomial $[e_3]$ is $n-3$.

Continuing the same process, we will finally obtain the polynomial [*e*] decomposed into *n* factors of the first degree, and of the same form. That is, the polynomial [*e*] will become equal to a product of *n* factors in the following manner:

$$x^n + Ax^{n-1} + Bx^{n-2} + \dots + Hx + K = (x-h)(x-i)(x-l) \dots\dots (x-t).$$

From the supposition, therefore, that the polynomial [*e*] is equal to zero, it follows that it may be decomposed as above. Now from the last formula it is evident that substituting for x any of the n values h, i, l, ... t, the equation $[e] = 0$ is fulfilled. Hence, the roots of the equation of the n^{th} degree are n in number.

Connection between the roots and the coefficients of any determined equation.

§ 99. The product of n equal factors of the form $(x-h)$ is (69, 70)

$$(x-h)^n = x^n - nhx^{n-1} + \frac{n(n-1)}{2}h^2x^{n-2} - \dots \pm h^n.$$

In the last equation of the preceding number, we have a product of n binomials; in which, however, the second term is different in each of them; but since the first term x is the same in all, the product of those n binomials with regard to x, must be equivalent to that of n factors, all equal to the same binomial $x-h$. That is, x will commence with the highest degree n, and orderly diminish it till the lowest possible degree. The difference of the two products must be in the coefficients of x, which, when all the binomials are equal to $x-h$, are nh, $\frac{n(n-1)}{2}h^2$. ... But the coefficients of the various powers of x, as well as the last term of the product, are formed in the same manner in both cases. That is, when the term subtracted from x is the same h in all the binomials, it is repeatedly used as a factor as many times and in the same manner as the different terms h, i, l. ... in the other case. It is, besides, to be observed, that when the terms subtracted from x are all unequal, each of them must have an equal share in the formation of the coefficients and last term of the product.

We may now institute an analytical comparison between the coefficients $-nh + \frac{n(n-1)}{2}h^2$, ... together with the last term h^n, when the factors are all equal, and those which are produced when the terms subtracted from x are different in all the binomials.

The first of these coefficients $-nh$ shows that $-h$ is used n times as a factor of x^{n-1}. But when the subtractive terms are all different, no term can be found in the product multiplied by $-h$ more than once, but all the same subtractive terms concur in like manner to form the coefficients; therefore, the coefficient equivalent to $-nh$ must be in this case $-(h+i+l+\dots+t)$; that is, the negative sum of all the roots of $[e]=0$. The next coefficient, or coefficient of the third term in the supposition of all the binomials being equal, is $+\frac{n(n-1)}{2}h^2$, which is $\frac{n(n-1)}{2}$ times the square of h. Now in the case of the terms taken from x, being all unequal, h can never multiply itself, and h^2 must be necessarily changed into the product of two different terms, for instance, hi; but again, all the terms taken from x, concur in an equal manner to the formation of the coefficient; and as in the coefficient $\frac{n(n-1)}{2}h^2$, the square of h is taken $\frac{n(n-1)}{2}$ times, the products of the terms, taken two and two, ought to be as many in number; and in fact, the number of combinations of n symbols, taken two and two, is (74) $\frac{n(n-1)}{2}$. Hence, the coefficient of the third term is $(hi+hl+\dots+ht+il+\dots it+\dots)$; that is, the positive sum of the products of all the roots of $[e]=0$, taken two and two. It is now easy to see, in the same manner, that the coefficient of the fourth term is the negative sum of the products of the same roots, taken three and three; the next, the positive sum of the products of all the roots, taken four and four, and so on. And the last term is the product of all the roots; a positive product if n is an even number, and a negative product if n is an uneven number. Our equation, therefore,

$$x^n+Ax^{n-1}+Bx^{n-2}+\dots+K=(x-h)(x-1)\dots(x-t)$$

is equivalent to

$$x^n+Ax^{n-1}+Bx^{n-2}+\dots+K=x^n-(h+i+\dots+t)x^{n-1}$$
$$+(hi+hl+\dots+ht+il+\dots)x^{n-2}+\dots\pm(h.i.l\dots.t),$$

in whatsoever manner x be taken. But (95, Th. 4, cor.) when two such polynomials are found equal to each other with any value of x, the corresponding coefficients of the same x are respectively equal to each other. So we will have

$$\left.\begin{array}{l}
A=-(h+i+l+\dots+t)\\
B=(hi+hl+\dots+ht+il+\dots+it+\dots)\\
C=-(hil+hig+\dots+hit+hlg+\dots+hlt+\dots)\\
\quad\text{\&c.}\\
K=\pm h.i.l\dots\dots t;
\end{array}\right\}(r).$$

That is, the coefficient of the second term of $[e] = 0$ is the negative sum of all the roots of the same equation; the coefficient of the following term is the positive sum of the products of the roots taken two and two, &c.

Corollaries.

§ 100. From this mutual connection between the roots and the coefficients of the equation $[e] = 0$, we infer some corollaries:

Corollary 1.

Case of one or more roots equal to zero.

If one of the roots should be equal to zero, the last term K of $[e]$ must be also equal to zero; and if two of the roots are equal to zero, the coefficient H also of the term before the last is equal to zero, for it contains the products of all the roots taken $(n - 1)$ and $(n - 1)$. In each one, therefore, of these products, there must be at least one of the roots equal to zero, and the whole coefficient is consequently equal to zero. Let the same be said of the coefficient preceding the two last terms, when three roots are equal to zero, and so on.

Corollary 2.

Signs of the roots changed.

By changing the signs of all the roots, the sign of the second term of $[e]$ will be also changed; that of the third will remain unvaried; the sign of the fourth will be changed; that of the fifth will remain as it is, &c. The reason of this is, that by changing the sign of all the factors, the signs of the products will be changed only when the number of factors is an uneven number.

Corollary 3.

Multiplying each one of the roots $h, i, l, \ldots . t$ of the equation $[e] = 0$ by the same quantity a, the coefficients A, B, C, ... H, and the last term K of $[e]$, will then become $a\mathrm{A}, a^2\mathrm{B}, a^3\mathrm{C}, \ldots . a^{n-1}\mathrm{H}, a^n\mathrm{K}$. Hence, *vice versâ*, if the terms of the equation

$$x^n + \mathrm{A}x^{n-1} + \mathrm{B}x^{n-2} + \ldots + \mathrm{H}x + \mathrm{K} = 0,$$

be orderly multiplied by the terms of the series

$$1, a, a^2, \ldots . a^{n-1}, a^n,$$

Roots multiplied.

that is, the first by the first, the second by the second, &c., the resulting polynomial

$$x^n + a\mathrm{A}x^{n-1} + a^2\mathrm{B}x^{n-2} + \ldots . + a^{n-1}\mathrm{H}x + a^n\mathrm{K},$$

made equal to zero, will be resolved with the same roots of $[e] = 0$. each one of them being multiplied by a.

Denominators eliminated.

Hence, also, if the coefficients of a given equation contain denominators, they may be all cleared of them without giving any coefficient to the first term. Suppose, in fact, that the different denominators are b, c, d, . . . Multiplying the terms of the given equation by the terms of the series 1, $(bc\ldots)$, $(bc\ldots)^2$, . . . the coefficients will be evidently all cleared of their denominators, while the first term of the equation remains unchanged. The roots, however, of the equation thus modified to be reduced to those of the given equation, must be divided by the product $(b.c\ldots)$

Corollary 4. The fourth corollary deserves to be particularly noticed, on account of its use in the resolution of the equations of the third and fourth degree.

In the equation $[e] = 0$, that is,

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots + Hx + K = 0,$$

x stands to represent any of the n roots h, i, . . . of the equation, which roots depend on the coefficients A, B, . . . in the manner above seen. Let us now suppose another equation of the same degree and form as $[e] = 0$, whose roots are $h + a$, $i + a$, &c.; that is, the same roots as $[e] = 0$, but each increased by the quantity a. We may represent this new equation as follows:

$$x'^n + A'x'^{n-1} + B'x'^{n-2} + \ldots + H'x' + K',$$

x' standing to represent any of the roots $h + a$, $i + a$, ... and as the coefficient A of $[e] = 0$ is the negative sum of all the roots h, i, l, . . . of that equation,

$$A' = -((h + a) + (i + a) + \ldots. + (t + a)),$$

Or, $$A' = -(h + i + l + \ldots. + t) - na,$$

But $$-(h + i + l + \ldots + t) = A,$$

therefore, $$A' = A - na.$$

In other words, changing the roots of $[e] = 0$ from x into $x' = x + a$, the coefficient A' of the second term of the new equation must be changed from A into $A' = A - na$.

Let us now suppose a to be taken equal to $\frac{A}{n}$; in this case $A' = A - A = 0$. That is, when the roots of the equation $[e] = 0$ are changed from x into $x' = x + \frac{A}{n}$, the new equation must be without the second term. And this equation being resolved, it will be enough to subtract from the different values of x' or roots the constant quantity $\frac{A}{n}$, to obtain the roots of the former equation.

The sums of various powers of the roots can be known, although the roots themselves are unknown.

§ 101. Call S_1 the sum of all the simple roots of $[e] = 0$, S_2 the sum of all the squares of the same roots, S_3 the sum of all the cubes, &c. We will have

$$S_1 = h + i + l + \dots\dots + t,$$
$$S_2 = h^2 + i^2 + l^2 + \dots\dots + t^2,$$
$$S_3 = h^3 + i^3 + l^3 + \dots\dots + t^3, \text{ \&c.}$$

Now, although the roots $h, i, \dots t$ may all remain unknown, yet the sums $S_1, S_2 \dots$ may be made known by the coefficients $A, B \dots$ of the equation.

With regard to the first, it is well known that the negative sum of all the roots is equal to the coefficient A of the second term. Hence,

$$S_1 = -A.$$

But to demonstrate the proposition with regard to all the sums, observe, first, that (98)

$$[e] = (x - h)[e_1],$$

But $\quad [e] = x^n + Ax^{n-1} + Bx^{n-2} + \dots + Hx + K,$

$$(x - h)[e_1] = x^n + (B_1 - h)x^{n-1} + (C_1 - hB_1)x^{n-2} + \dots + (K_1 - hH_1)x - hK_1.$$

Now, since the two first members of these equations are equal to each other for any value of x, so also are the second members. Hence, according to the fifth theorem (95),

$$A = B_1 - h,\ B = C_1 - hB_1,\ \dots\dots\ H = K_1 - hH,\ K = -hK_1.$$

From which we infer

$$B_1 = A + h,$$
$$C_1 = B + hB_1 = B + hA + h^2,$$
$$D_1 = C + hC_1 = C + hB + h^2A + h^3,$$
$$\text{\&c.,}$$
$$K_1 = H + hG + \dots + h^{n-2}A + h^{n-1}.$$

But from $[e] = (x - h)[e_1]$ we have $\dfrac{[e]}{x - h} = [e_1]$ and (98) $[e_1] = x^{n-1} + B_1x^{n-2} + \dots + H_1x + K_1$. Hence,

$$\frac{[e]}{x - h} = [e_1] = x^{n-1} + (A + h)x^{n-2} + (B + hA + h^2)x^{n-3} + (C + hB + h^2A + h^3)x^{n-4} + \dots\dots + H + hG + \dots\dots + h^{n-2}A + h^{n-1}.$$

And since what we say of h may be equally said of any other root, we will have in equal manner,

$$\frac{[e]}{x-i} = x^{n-1} + (A+i)x^{n-2} + (B+iA+i^2)x^{n-3} + \ldots . + H + iG$$
$$+ \ldots . + i^{n-2}A + i^{n-1},$$

and so on.

Now the roots of $[e_1] = 0$ are the same as those of $[e] = 0$, with the exception of h; hence, the roots of $\frac{[e]}{x-h} = 0$ are the same as the roots of $[e] = 0$, with the exception of h. Hence, also, on account of the well-known dependence of the roots on the coefficients of the equation,

$$-(A+h) = i + l + \ldots . + s + t \,\}(c_1).$$

Reasoning in the same manner with regard to the equations $\frac{[e]}{x-i} = 0, \ldots . \frac{[e]}{x-t} = 0$, we will have

$$\left.\begin{aligned} -(A+i) &= h + l + \ldots . + s + t \\ &\;\vdots \\ -(A+t) &= h + i + \ldots . + s \end{aligned}\right\}(c_1).$$

The coefficient B in $[e]$ is equal to the sum of the products of the roots, taken two and two. If we suppose one of the roots wanting, for instance h, the products of the remaining roots, taken two and two, will be given by $B - h(i + l + \ldots . + t)$. Now the coefficient of the third term of $\frac{[e]}{x-h} = 0$ is equal to the sum of the products of the roots of $[e] = 0$, with the exception of h, taken two and two. Hence, $B + hA + h^2 = B - h(i + l + \ldots t)\,\}(c_2)$.

In like manner from the equations $\frac{[e]}{x-i} = 0 \ldots . \frac{[e]}{x-t} = 0$, we have

$$\left.\begin{aligned} B + iA + i^2 &= B - i(h + l + \ldots + t) \\ &\;\vdots \\ B + tA + t^2 &= B - t(h + i + \ldots + s) \end{aligned}\right\}(c_2).$$

The products of the n roots, taken three and three, are given by $-$ C; that is, by the coefficient of the fourth term of $[e]$ taken with an opposite sign. But supposing the root h to be taken from the number of the n roots, the sum of the products of the remaining roots, taken three and three, will be $-C - h(il + \ldots + it + \ldots + st)$. Now this very sum is given by the coefficient of the fourth term of $\frac{[e]}{x-h}$, taken with an opposite sign. Hence,

$$-(C + hB + h^2A + h^3) = -C - h(il + \ldots + it + \ldots + st)\}(c_3).$$

And from the equations $\frac{[e]}{x - i} = 0 \ldots \frac{[e]}{x - t} = 0$,

$$\left.\begin{array}{l} -(C + iB + i^2A + i^3) = -C - i(hl + \ldots + ht + \ldots + st) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ -(C + tB + t^2A + t^3) = -C - t(hi + \ldots + ht + \ldots\ldots) \end{array}\right\}(c_3).$$

&c.

Now the sum of all the first members of (c_1) is $-nA - S_1$; the sum of the first members of (c_2) is $nB + AS_1 + S_2$; the sum of the first members of (c_3) is $-nC - BS_1 - AS_2 - S_3$, &c... But the sum of the first members is equal to the sum of the corresponding second members; and with regard to the second members of (c_1), observe, that if to each one of these second members, we add one of the n roots in this manner: h to the second member of the first equation, i to the second member of the second, and so on, and finally, t to the second member of the last or nth equation, this addition will make the second members contain the sum of all the roots of $[e] = 0$ or $-A$, and all the n second members equal to each other, and their sum equal to $-nA$; if, therefore, from this sum we subtract $-A$, we will have the sum of the second members without the above addition, which is $-nA + A$ or $A(1 - n)$. And therefore, since we have found $-nA - S_1$ for the sum of the corresponding first members, $-nA - S_1 = A(1 - n)$, or

$$S_1 + nA = A(n-1).$$

The second members of the following (c_2), (c_3) &c... contain two parts; the first of which gives evidently for sums, nB, $-nC$.... To obtain the sum of the latter part with regard to (c_2), remark that each of the n roots $h, i, \ldots t$, is in the second members successively a factor of all the others; therefore, when h is a factor we have $hi, hl, \ldots\ldots$ and when i and $l \ldots$ are factors, we will have $ih, lh \ldots$; that is, in the whole sum of the latter part of (c_2), each one of the products of the n roots, taken two and two, will appear twice. Now B gives the sum of the products of the n roots taken two and two; therefore, the sum of the latter part of (c_2) is $-2B$, which added to the sum nB of the first part, gives $nB - 2B$ for the whole sum of the second members of (c_2). But the sum of the corresponding first members is $nB + AS_1 + S_2$; therefore,

$$S_2 + AS_1 + nB = B(n-2).$$

In the latter part of the equations (c_3) we may likewise observe that each of the n roots becomes in succession a factor of all the other roots, taken two and two; and in the same manner as h multiplies il, i multiplies hl, and l, hi, so that the product hil is to be found three times in the sum, as also are all the others. Hence the whole sum of the latter part of the second member is in this case three times the products of n roots, taken three and three. Now the n roots, taken positively three and three, are expressed by $-C$; therefore, the same sum being negative, will be given by $+3C$, which added to the sum $-nC$ of the first part, gives $-nC+3C$, or $-C(n-3)$, for the whole sum of the second members of (c_3). But the sum of the corresponding first members is $-nC-BS_1-AS_2-S_3$; therefore,

$$S_3+AS_2+BS_1+nC=C(n-3), \text{ \&c.}$$

From this, and from the two preceding equations, we easily infer the values of S_1, S_2, S_3 as follows:

$$\begin{aligned}
S_1 &= -A,\\
S_2 &= -AS_1-2B = A^2-2B,\\
S_3 &= -AS_2-BS_1-3C = -A^3+3AB-3C, \text{ \&c.};
\end{aligned}$$

that is, the sums S_1, S_2, S_3 . . . of the various powers of the roots h, i, l, . . . t, are given by the known coefficients A, B, C . . . . of the equation $[e]=0$, whether the roots themselves be known or not.

Corollary and criterion.

Since A^2-2B gives the sum of all the squares of the roots of the equation $[e]=0$, in the supposition that all the roots are real, the binomial A^2-2B cannot be but a positive quantity, for it is equal to a sum of terms, all of them essentially positive. But the sign of A^2-2B depends on the values of A and B as they are to be found in $[e]$; if, therefore, the sign of this difference is negative, it is certain that not all the roots of $[e]=0$ are real.

Conjugate imaginary roots.

§ 102. The imaginary expressions

$$u+v\sqrt{-1},\ u-v\sqrt{-1}$$

are called *conjugate*, for $[e]=0$ cannot have for one of its roots an expression of the form of one of the two conjugates, without having at the same time also the other.

Before we demonstrate this proposition, it is to be observed that

$$(a+b\sqrt{-1})^2=a^2-b^2+2ab\sqrt{-1},$$
$$(a-b\sqrt{-1})^2=a^2-b^2-2ab\sqrt{-1};$$

and making $a^2 - b^2 = h$, $2ab = K$,

$$(a + b\sqrt{-1})^2 = h + K\sqrt{-1},$$
$$(a - b\sqrt{-1})^2 = h - K\sqrt{-1},$$

from these,

$$(a + b\sqrt{-1})^3 = (a + b\sqrt{-1})(h + K\sqrt{-1}),$$
$$(a - b\sqrt{-1})^3 = (a - b\sqrt{-1})(h - K\sqrt{-1}).$$

Now (68) $(a \pm b\sqrt{-1})(h \pm K\sqrt{-1}) = (ah - bK) \pm (aK + bh)\sqrt{-1}$.

And making $ah - bK = l$, $aK + bh = m$;

$$(a \pm b\sqrt{-1})(h \pm K\sqrt{-1}) = l \pm m\sqrt{-1},$$

Hence,

$$(a + b\sqrt{-1})^3 = l + m\sqrt{-1},$$
$$(a - b\sqrt{-1})^3 = l - m\sqrt{-1}.$$

In like manner, we have

$$(a + b\sqrt{-1})^4 = n + 0\sqrt{-1},$$
$$(a - b\sqrt{-1})^4 = n - 0\sqrt{-1},$$

and generally

$$(a + b\sqrt{-1})^m = A + B\sqrt{-1},$$
$$(a - b\sqrt{-1})^m = A - B\sqrt{-1}.$$

The equation [e] = 0 cannot admit of one of the conjugates without admitting also of the other.

§ 108. In the supposition that one of the roots of [e] = 0 has the imaginary form $u + v\sqrt{-1}$, substituting in [e] = 0, that is, in $x^n + Ax^{n-1} + \ldots\ldots + K = 0$, that value instead of x, the equation will take the form

$$U + V\sqrt{-1} = 0;$$

for taking separately each term of the equation, we will have

$$\begin{aligned} x^n &= (u + v\sqrt{-1})^n &&= u' + v'\sqrt{-1}, \\ Ax^{n-1} &= A(u + v\sqrt{-1})^{n-1} &&= u'' + v''\sqrt{-1}, \\ Bx^{n-2} &= \ldots\ldots\ldots\ldots &&= u''' + v'''\sqrt{-1}, \text{ \&c.} \\ K &= K. \end{aligned}$$

And, consequently, calling U the sum of the terms u', u'' .... K and V the sum of the coefficients v', v'' .... $v^{(n)}$, we will have

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots + K = U + V\sqrt{-1}.$$

Now $U + V\sqrt{-1}$ cannot be equal to zero, unless separately U and V are each equal to zero; because U, a real term, can never be eliminated by $V\sqrt{-1}$, an imaginary one; hence, $[e] = 0$, which, in our supposition, is $U + V\sqrt{-1} = 0$, necessarily supposes $U = 0$ and $V = 0$. It is now easy to see that when $x = u + v\sqrt{-1}$ is a root of $[e] = 0$, $x = u - v\sqrt{-1}$ is a root of the same equation likewise. Because, substituting this value of x in each term of $[e]$, we have,

$$\begin{aligned} x^n &= (u - v\sqrt{-1})^n = u' - v'\sqrt{-1}, \\ Ax^{n-1} &= A(u - v\sqrt{-1})^{n-1} = u'' - v''\sqrt{-1}, \\ Bx^{n-2} &= \ldots\ldots\ldots = u''' - v'''\sqrt{-1}, \text{ \&c.} \\ K &= K. \end{aligned}$$

And therefore,

$$x^n + Ax^{n-1} + \ldots\ldots + K = U - V\sqrt{-1}.$$

But when $x = u + v\sqrt{-1}$ is a root of $[e] = 0$, U and V are separately each equal to zero; hence, $U - V\sqrt{-1}$, as well as $U + V\sqrt{-1}$, is equal to zero. But $U - V\sqrt{-1}$ is that which $[e]$ becomes when $u - v\sqrt{-1}$ is substituted for x; hence, $x = u - v\sqrt{-1}$ is a root of $[e] = 0$. Therefore, when one of the conjugate radical expressions is a root of the equation, the other also is necessarily a root of the same equation.

Corollaries. From this connection it follows, first, that the number of the roots of the imaginary form $(a \pm b\sqrt{-1})$ must necessarily be even.

Secondly, since whatever be the roots of $[e] = 0$, we have always

$$[e] = (x - h)(x - i) \ldots\ldots (x - s)(x - t).$$

Supposing that the first two, or four, or eight and so on, are imaginary, we will have for example, $h = a + b\sqrt{-1}$, $i = a - b\sqrt{-1}$; hence,

$$(x - h)(x - i) = (x - a - b\sqrt{-1})(x - a + b\sqrt{-1}) = (x - a)^2 + b^2,$$

and, consequently,

$$[e] = [(x - a)^2 + b^2](x - l) \ldots\ldots (x - t).$$

That is to say, whatever be the nature of the roots of $[e] = 0$, the polynomial $[e]$ is always capable of being decomposed into real factors, either of the first or of the second degree.

ARTICLE IV.

Resolution of Determined Equations of the Third and Fourth Degrees, having Real Coefficients.

General formulas of the equations of the third degree.

§ 104. A GENERAL formula expressing any equation of the third degree, may be as follows:

$$x^3 + Ax^2 + Bx + C = 0 \quad (r).$$

Now we have seen (100) that equations of any degree can be cleared of the second term, and (r) can become

$$x^3 + Hx + K = 0 \quad (r').$$

Which being resolved, we may obtain the roots of r, by taking $\frac{A}{3}$ from each of the roots of (r'), for (r') is deduced from (r) by substituting x' or $x + \frac{A}{3}$ instead of x.

Now, from $x' = x + \frac{A}{3}$, we have also $x = x' - \frac{A}{3}$, which, if substituted in (r), will give us the equation,

$$x'^3 + Hx' + K = 0,$$

in which x' is the same as the x of (r) and $H = B - \frac{A^2}{3}$, $K = \frac{A^3}{9} - \frac{A^3}{27} - \frac{AB}{3} + C$. But when the coefficient H and the term k are thus determined, it is immaterial to call the variable either x or x', since the roots must be such as to correspond to H and K; we may therefore use (r') as well as the last equation.

Observe also, that the formula (r') is as general as (r); and since the resolution of (r') gives the resolution of (r) also, all that we may say with regard to the resolution of (r') can be applied to the resolution of any equation of the third degree.

Roots of the general equation of the third degree.

§ 105. Since the degree of the equation is an uneven number, the equation $(r') = 0$ contains certainly (96) one real root at least; the other two will be (103) either both real or both imaginary. Calling, therefore, h the real root, and the other two i and l; the equation $(r') = 0$ will be (98) equivalent to

$$(x - h)[(x - i)(x - l)] = 0,$$

in which $(x - i)(x - l) = x^2 - (i + l)x + il.$

Again, (r') does not contain the second term, which supposes (99 . r) equal to zero, the sum of the roots of $(r') = 0$; that is, $h + i + l = 0$; and consequently, $h = -(i + l)$.

To find out the quality of the roots i and l, make $-(i + l) = 2a$ and $il = a^2 - 3c$; or, which is the same, make

$$h = 2a, \ (x - i)(x - l) = x^2 + 2ax + a^2 - 3c,$$

which values being substituted in (r'), or

$$(x - h)[(x - i)(x - l)] = 0,$$

we will have $$(x - 2a)(x^2 + 2ax + a^2 - 3c) = 0,$$

or $$x^3 - 3(a^2 + c)x + 6ac - 2a^3 = 0;$$

and consequently,

$$\left.\begin{array}{l} \mathrm{H} = -3(a^2 + c) \\ \mathrm{K} = \quad 6ac - 2a^3 \end{array}\right\} (f),$$

Resolving now the equation

$$(x - i)(x - l) = x^2 + 2ax + (a^2 - 3c) = 0,$$

we have (93) $$i = -a + \sqrt{3c},$$
$$l = -a - \sqrt{3c},$$

which are either real or imaginary, according as c is positive or negative. Now from H and K that are given, and from the equations (f), we may find out whether c is positive or negative.

The equations (f) may be changed as follows:

$$\frac{\mathrm{H}}{3} = -a^2 - c, \ \frac{\mathrm{K}}{2} = 3ac - a^3,$$

from which $$\frac{\mathrm{H}^3}{27} = -a^6 - 3a^4c - 3a^2c^2 - c^3,$$
$$\frac{\mathrm{K}^2}{4} = 9a^2c^2 - 6a^4c + a^6;$$

and consequently,

$$\begin{aligned} \frac{\mathrm{K}^2}{4} + \frac{\mathrm{H}^3}{27} &= -9a^4c + 6a^2c^2 - c^3, \\ &= -c(9a^4 - 6a^2c + c^2), \\ &= -c(3a^2 - c)^2. \end{aligned}$$

Now $(3a^2 - c)^2$ is essentially positive. When, therefore, $\frac{\mathrm{K}^2}{4} + \frac{\mathrm{H}^3}{27}$ is positive, the factor c of the second member must be negative, and when the same binomial is negative, the factor c must be positive.

But when c is positive, all the roots of $(r') = 0$ are real. Hence, the roots of the equation,

$$x^3 + Hx + K = 0,$$

are all real when the cube $\left(\frac{H}{3}\right)^3$ of one-third of the coefficient of x, plus the square of one half of the last term, give a negative sum; if the sum is positive, then two of the roots of $(r') = 0$ are imaginary.

Let us apply the criterion to the following examples:

$$(1.)\quad x^3 - 3x + 52 = 0,$$
$$(2.)\quad x^3 - 19x + 30 = 0,$$

from the first in which $H = -3$, $K = 52$, we have

$$\frac{K^2}{4} + \frac{H^3}{27} = 676 - 1 = +675.$$

The sum is positive; therefore, two of the roots of (1) are imaginary; and, in fact, the roots of this equation are

$$x = -4,\ x = 2 + 3\sqrt{-1},\ x = 2 - 3\sqrt{-1}.$$

From the second in which $H = -19$, $K = 30$, we have

$$\frac{K^2}{4} + \frac{H^3}{27} = 225 - \frac{6859}{27} = -\frac{784}{27}.$$

The sum is negative; therefore, the roots of (2) are real, and in fact, the roots of this equation are

$$x = 2,\ x = 3,\ x = -5.$$

Resolution of the same general equation.

§ 106. It remains now for us to see in what manner these roots, either real or imaginary, may be found and determined.

Two conditions required.

And here observe, that to have any quantity exactly determined, it is not enough to have it explicitly given by a function of other quantities which are known; but it is necessary, besides, that the function itself be reducible to a determined and explicit value. Thus, for example, in the equation

$$x = \sqrt{20},$$

we have the unknown quantity x explicitly given by a function of a known quantity. But this function can never be exactly determined, for $\sqrt{20}$ is an irrational number. And more generally the unknown quantity explicitly given by any function of known quantities follows the nature of the function; and in cases in which the value of the function could not be determined, either exactly or in any way, the unknown quantity also would remain undetermined or altogether unknown.

The first condition always fulfilled, but not the second.

Now with regard to the resolution of our general equation, we can always obtain the values of the roots explicitly given by a function of H and K, which are known quantities; but the function itself is not reducible to a definite term, except in some cases.

Let us first see how the first condition is always verified.

Take with the general equation,

$$x^3 + Hx + K = 0 \quad (r'),$$

the other of the second degree,

$$z^2 + Kz - \left(\frac{H}{3}\right)^3 = 0 \ (r''),$$

having for the coefficient of z the last term of (r'), and for the last term the cube of one-third of the coefficient of x in (r'). Now the equation (r'') can be resolved, and calling z_1, z_2 its roots, the roots also of (r') will be given by the different values of the binomial

$$z_1^{\frac{1}{3}} + z_2^{\frac{1}{3}}.$$

In fact, the equation $(r') = 0$ is fulfilled when the binomial $z_1^{\frac{1}{3}} + z_2^{\frac{1}{3}}$ is substituted instead of x. To see this, make $z_1^{\frac{1}{3}}\, u$, $z_2^{\frac{1}{3}} = v$, or

$$z_1^{\frac{1}{3}} + z_2^{\frac{1}{3}} = u + v.$$

and the substitution of this binomial being made in (r_1) we will have

$$(u + v)^3 + H(u + v) + K.$$

Now from the equation (r'') we have (99)

$$K = -z_1 - z_2 = -u^3 - v^3,$$

$$-\left(\frac{H}{3}\right)^3 = z_1 z_2 = u^3 v^3;$$

that is,

$$H = -3uv,$$

and substituting these values of H and K in the last formula, it will become

$$(u + v)^3 - 3uv(u + v) - (u^3 + v^3),$$

which, if $(u + v)$ is a root of $(r') = 0$, must be equal to zero. Now evolving the first and second terms of this trinomial, we have

$$u^3 + 3u^2v + 3uv^2 + v^3 - 3u^2v - 3uv^2 - u^3 - v^3 = 0.$$

Hence, the binomial $u + v$ or $z_1^{\frac{1}{3}} + z_2^{\frac{1}{3}}$ substituted in (r') fulfils the equation, and $z_1^{\frac{1}{3}} + z_2^{\frac{1}{3}}$ is a root of the equation.

But $z_1^{\frac{1}{3}} + z_2^{\frac{1}{3}}$ admits of different values, some of which must be excluded. That is, all those values, and only those, which make $-(u^3 + v^3) = \mathrm{K}$ and $-3uv = \mathrm{H}$, will make also $u + v = 0$ a root of (r').

From the equations $u = z_1^{\frac{1}{3}}$, $v = z_2^{\frac{1}{3}}$, we have also

$$u = z_1^{\frac{1}{3}} \sqrt[3]{1}; \quad v = z_2^{\frac{1}{3}} \sqrt[3]{1};$$

but $\sqrt[3]{1}$ has the following different values:

$$\sqrt[3]{1} = 1,$$

$$\sqrt[3]{1} = \frac{-1 + 3^{\frac{1}{2}}\sqrt{-1}}{2},$$

$$\sqrt[3]{1} = \frac{-1 - 3^{\frac{1}{2}}\sqrt{-1}}{2},$$

because each of them, raised to the third power, gives 1.

Therefore, u and v admit each of three different values; that is, the three values of u, are

$$z_1^{\frac{1}{3}}, \; z_1^{\frac{1}{3}}\left[\frac{-1 + 3^{\frac{1}{2}}\sqrt{-1}}{2}\right], \; z_1^{\frac{1}{3}}\left[\frac{-1 - 3^{\frac{1}{2}}\sqrt{-1}}{2}\right]$$

and the three values of v,

$$z_2^{\frac{1}{3}}, \; z_2^{\frac{1}{3}}\left[\frac{-1 + 3^{\frac{1}{2}}\sqrt{-1}}{2}\right], \; z_2^{\frac{1}{3}}\left[\frac{-1 - 3^{\frac{1}{2}}\sqrt{-1}}{2}\right].$$

Now among these values those only may be used from which we obtain $-(u^3 + v^3) = \mathrm{K}$, $-3uv = \mathrm{H}$. The term K will be always obtained in the same manner, whatever be the values chosen for u and v; since, in all cases $u^3 + v^3 = z_1 + z_2$ and $-(z_1 + z_2) = \mathrm{K}$; but with regard to H, not all the values of u and v can give it, but those only whose product is $z_1^{\frac{1}{3}} z_2^{\frac{1}{3}}$. Now this product is obtained in the three following manners only: Multiplying the first value of u by the first of v; the second value of u by the third of v, and the third value of u by the second of v. The roots, therefore, x_1, x_2, x_3 of $(r') = 0$, will be represented as follows:

$$x_1 = z_1^{\frac{1}{3}} + z_2^{\frac{1}{3}},$$

$$x_2 = \frac{-1 + 3^{\frac{1}{2}}\sqrt{-1}}{2} z_1^{\frac{1}{3}} + \frac{-1 - 3^{\frac{1}{2}}\sqrt{-1}}{2} z_2^{\frac{1}{3}},$$

$$x_3 = \frac{-1 - 3^{\frac{1}{2}}\sqrt{-1}}{2} z_1^{\frac{1}{3}} + \frac{-1 + 3^{\frac{1}{2}}\sqrt{-1}}{2} z_2^{\frac{1}{3}}.$$

These are the expressions of the three roots of the general equation (r') of the third degree, in which the coefficients of $z_1^{\frac{1}{3}}$ and $z_2^{\frac{1}{3}}$ are either equal to unity or of an imaginary form. With regard to z_1 and z_2, which are the roots of $(r'') = 0$, we have their values (93), as follows:

$$z_1 = -\frac{K}{2} + \sqrt{\left(\frac{H}{3}\right)^3 + \frac{K^2}{4}},$$

$$z_2 = -\frac{K}{3} - \sqrt{\left(\frac{H}{3}\right)^3 + \frac{K^2}{4}},$$

imaginary or real, accordingly as the binomial under the radical sign is either negative or positive. But from the criterion given in the last number, when this same binomial is negative, the roots of $(r') = 0$ are all real, and when the binomial is positive, two of the roots of $(r') = 0$ are imaginary. That is, when the roots of (r'') are imaginary, all the roots of (r') are real; and when the roots of (r'') are real, two of the roots of (r') are imaginary. Again, whenever the binomial $\left(\frac{H}{3}\right)^3 + \frac{K^2}{4}$ is not equal to zero, and all the roots of $(r') = 0$ are real, they are exclusively given by terms and factors of an imaginary form.

From all this, it follows that the roots of the equation (r') may be always given by explicit functions of the known terms H and K, and the first of the two conditions is, consequently, fulfilled in all cases. But we will see, by some examples, that the functions themselves are not always reducible to explicit and definite values, which is the second condition to be fulfilled to have the equation (r') completely resolved.

In the case of the binomial $\left(\frac{H}{8}\right)^3 + \frac{K^2}{4} = 0$, the roots of (r'') are real and equal to each other; namely,

$$z_1 = z_2 = -\frac{K}{2}.$$

The roots also of (r') are all real, and two of them equal to each other; that is,

$$x_1 = -2\left(\frac{K}{2}\right)^{\frac{1}{3}},$$

$$x_2 = x_3 = \left(\frac{K}{2}\right)^{\frac{1}{3}};$$

and consequently, $$-x_1 =_2 x + x_3.$$

Examples. § 107. Given equations:

(1.) $x^3 - 6x^2 + 3x + 20 = 0.$

(2.) $x^3 + 3x - 14 = 0.$

(3.) $x_3 - 12x + 16 = 0.$

The first of these equations is to be cleared of the second term, which is easily done by substituting (104) $x' + \frac{6}{3}$, or $x' + 2$, instead of x. In this manner, we will have

$$(x'+2)^3 - 6(x'+2)^2 + 3(x'+2) + 20 = 0,$$

or
$$x'^3 - 9x' + 10 = 0,$$

containing the roots of the given equation (1), but diminished each by the number 2; for from $x = x' + 2$, it follows that $x' = x - 2$. Hence, after having found the roots of the last equation, it is enough to add to each of them the number 2, to have the roots of the given equation (1). Now, to resolve the last equation, let us compare it with the general equation (r'), and we will have

$$H = -9,\ K = 10;$$

and therefore,
$$\frac{K}{2} = 5,\ \frac{H}{3} = -3;$$

hence,
$$\left(\frac{H}{3}\right)^3 + \frac{K^2}{4} = -2.$$

The binomial being negative, the roots of the equation are all real. And with regard to these roots, we have first, from the preceding number,

$$z_1 = -5 + \sqrt{-2},$$
$$z_2 = -5 - \sqrt{-2},$$

Hence, $(z_1)^{\frac{1}{3}} = (-5 + \sqrt{-2})^{\frac{1}{3}}$, $(z_2)^{\frac{1}{3}} = (-5 - \sqrt{-2})^{\frac{1}{3}}$, which are to be substituted in the values of the roots x_1, x_2, x_3. But before making this substitution, let us reduce $(z_1)^{\frac{1}{3}}$ and $(z_2)^{\frac{1}{3}}$ to a simpler form, as follows:

Make
$$\begin{aligned} -5 \pm \sqrt{-2} &= (y \pm \sqrt{-2})^3 \\ &= y^3 \pm 3y^2\sqrt{-2} - 6y \pm 2\sqrt{-2}, \\ &= y^3 - 6y \pm (3y^2 - 2)\sqrt{-2}, \end{aligned}$$

which comes to the same as to take

$$y^3 - 6y = -5,$$
$$3y^2 - 2 = 1.$$

Now from this last we have $y^2 = 1$, and consequently,

$$y = \pm 1.$$

But since the positive value of y alone substituted in $y^3 - 6y$ makes it equal to -5, therefore $+1$ is the only admissible value for y; hence,

$$-5 \pm \sqrt{-2} = (1 \pm \sqrt{-2})^3;$$

and consequently,

$$(-5 \pm \sqrt{-2})^{\frac{1}{3}} = 1 \pm \sqrt{-2},$$

and $$z_1^{\frac{1}{3}} = 1 + \sqrt{-2},\ z_2^{\frac{1}{3}} = 1 - \sqrt{-2};$$

hence, also $x_1 = 2,\ x_2 = -1 - \sqrt{6},\ x_3 = -1 + \sqrt{6};$

and consequently, adding 2 to each of these, we will have, for the roots of the given equation (1),

$$4,\ 1 - \sqrt{6},\ 1 + \sqrt{6}.$$

General remark. Let us remark here that since the last term of the equation is the product of the roots of the same equation (99 . *r*), we may succeed in finding the roots among the factors of the last term, by trying if any of them fulfils the equation. Thus, among the factors of the last term 20 of the preceding example (1), there is the number 4 which fulfils the equation; to find the other two, divide the equation by $x - 4$, and we will have $x^2 - 2x - 5 = 0$, which, resolved, gives $x = -1 \pm \sqrt{6}$.

The observation just made is general; that is, applicable to equations of any degree.

Example 2. The equation (2) does not contain the second term; and consequently we may immediately compare it with (r'), from which comparison, we have

$$H = 3,\ K = -14,$$

and $$\frac{K^2}{4} + \frac{H^3}{27} = 50.$$

The binomial is positive; therefore two of the roots of the equation are imaginary. With regard to z_1 and z_2, we will have

$$z_1 = 7 + \sqrt{50},$$
$$z_2 = 7 - \sqrt{50}.$$

and making $7 \pm 5\sqrt{2} = (y \pm \sqrt{2})^3$

$$= (y^3 + 6y) \pm (3y^2 + 2)\sqrt{2},$$

or, which is the same,

$$y^3 + 6y = 7,\ 3y^2 + 2 = 5,$$

we have from the last $y^2 = 1$;

that is, $y = \pm 1$,

but since $+1$ only fulfils the other equation,

$$y = 1;$$

and consequently, $7 \pm 5\sqrt{2} = (1 \pm \sqrt{2})^3$,

and $z_1^{\frac{1}{3}} = 1 + \sqrt{2},\ z_2^{\frac{1}{3}} = 1 - \sqrt{2}$.

Hence, $x_1 = 2,\ x_2 = -1 + \sqrt{6}\sqrt{-1},\ x_3 = -1 - \sqrt{6}\sqrt{-1}$,
for the roots of the equation (2).

Example 3.

The last equation (3), compared with (r'), gives

$$\mathrm{H} = -12,\ \mathrm{K} = 16;$$

hence, $\left(\frac{\mathrm{H}}{3}\right)^3 + \frac{\mathrm{K}^2}{4} = -64 + 64 = 0;$

therefore (106), $z_1 = z_2 = -8$,

and $x_1 = -4,\ x_2 = x_3 = 2$,

for the roots of the equation (3).

EQUATIONS OF THE FOURTH DEGREE.

Resolution of the equations of the fourth degree.

§ 108. The preceding method to resolve equations of the third degree is applicable, with some modifications, also to those of the fourth degree. The following,

$$x^4 + \mathrm{G}x^2 + \mathrm{H}x + \mathrm{K} = 0 \quad (q),$$

is the general formula of the equations of the fourth degree, cleared of the second term. To resolve it, take the equation,

$$z^3 + \frac{\mathrm{G}}{2}z^2\left(\frac{\mathrm{G}^2}{16} - \frac{\mathrm{K}}{4}\right)z - \frac{\mathrm{H}^2}{64} = 0 \quad (q')$$

of the third degree, together with (q), and let the roots of (q') be called z_1, z_2, z_3.

The roots of (q) will be given by the addition, either positive or negative, of $z_1^{\frac{1}{2}}$, $z_2^{\frac{1}{2}}$, $z_3^{\frac{1}{2}}$, and the difference between the same expressions, variously taken. To prove it, observe first, that (99 . r)

$$z_1 + z_2 + z_3 = -\frac{\mathrm{G}}{2},$$

$$z_1z_2 + z_1z_3 + z_2z_3 = \left(\frac{\mathrm{G}^2}{16} + \frac{\mathrm{K}}{4}\right)$$

$$z_1z_2z_3 = \left(\frac{\mathrm{H}}{8}\right)^2.$$

From which, taking

$$\pm z_1^{\frac{1}{2}} = u,\ \pm z_2^{\frac{1}{2}} = v,\ \pm z_3^{\frac{1}{2}} = w,$$

we have

$$\left.\begin{array}{l} u^2 + v^2 + w^2 = -\dfrac{G}{2} \\ u^2v^2 + u^2w^2 + v^2w^2 = \left(\dfrac{G}{4}\right)^2 - \dfrac{K}{4} \\ u^2v^2w^2 = \left(\dfrac{H}{8}\right)^2 \end{array}\right\} (q'');$$

and from these

$$G = -2u^2 - 2v^2 - 2w^2,$$
$$K = u^4 + v^4 + w^4 - 2u^2v^2 - 2u^2w^2 - 2v^2w^2,$$
$$H = 8uvw, \text{ or } H = -8uvw;$$

but let us take the signs of the factors u, v, w in such a manner as to have $H = -8uvw$.

Substituting now in (q) the values of G, H, K, given by the last equations, we have

$$x^4 - 2(u^2 + v^2 + w^2)x^2 - 8uvwx$$
$$+ u^4 + v^4 + w^4 - 2u^2v^2 - 2u^2w^2 - 2v^2w^2 = 0.$$

Now making in this equation (which does not differ from (q), except in form) $x = u + v + w$, the first member becomes zero, and the equation is resolved; x, therefore, equal to $u + v + w$, is the root of the equation (q).

But H is either positive or negative: in the first case we may have $H = -8uvw$, taking u, v, w in four different manners, as follows:

$$u = +\sqrt{z_1},\ v = +\sqrt{z_2},\ w = -\sqrt{z_3},$$
$$u = +\sqrt{z_1},\ v = -\sqrt{z_2},\ w = +\sqrt{z_3},$$
$$u = -\sqrt{z_1},\ v = +\sqrt{z_2},\ w = +\sqrt{z_3},$$
$$u = -\sqrt{z_1},\ v = -\sqrt{z_2},\ w = -\sqrt{z_3}.$$

Hence, when H is positive, the roots of the equation (q) are

$$x_1 = \sqrt{z_1} + \sqrt{z_2} - \sqrt{z_3}, \qquad x_2 = \sqrt{z_1} - \sqrt{z_2} + \sqrt{z_3},$$
$$x_3 = -\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}, \qquad x_4 = -\sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3}.$$

When H is negative, we may likewise have $H = -8uvw$ in four different manners, taking the values of u, v, w, as follows:

$$u = +\sqrt{z_1},\ v = +\sqrt{z_2},\ w = +\sqrt{z_3},$$
$$u = +\sqrt{z_1},\ v = -\sqrt{z_2},\ w = -\sqrt{z_3},$$

$$u = -\sqrt{z_1},\ v = -\sqrt{z_2},\ w = +\sqrt{z_3}$$
$$u = -\sqrt{z_1},\ v = +\sqrt{z_2},\ w = -\sqrt{z_3}$$

And the roots of the equation (q) are in this case,

$$x_1 = \sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3} \qquad x_2 = \sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3},$$
$$x_3 = -\sqrt{z_1} - \sqrt{z_2} + \sqrt{z_3}, \qquad x_4 = -\sqrt{z_1} + \sqrt{z_2} - \sqrt{z_3}$$

The addition, therefore, either positive or negative, of $z_1^{\frac{1}{2}}$, $z_2^{\frac{1}{2}}$, $z_3^{\frac{1}{2}}$, and the difference of the same expressions, give the roots of the equation (q).

Quality of the roots: how found out.

It is now evident that when $\sqrt{z_1}$, $\sqrt{z_2}$, $\sqrt{z_3}$ are real expressions, their sum, either positive or negative, and their difference in whatever manner it is taken, will always give a real result, and consequently, real roots. On the contrary, when one or more of the radicals are imaginary, the same results from addition and from taking their difference will be likewise imaginary, unless the imaginary terms be mutually destroyed. Hence, to know when the roots of (q), or at least some of them, are real and when imaginary, it is enough to know whether the radicals $\sqrt{z_1}$, $\sqrt{z_2}$, $\sqrt{z_3}$ are real or imaginary.

Observe, now, that from the third of the equations (q''), we have

$$z_1 \,.\, z_2 \,.\, z_3 = +\left(\frac{\mathrm{H}}{8}\right)^2;$$

that is to say, the roots of (q') give a positive product; but the positive product of three factors cannot be obtained, unless one of them is positive, and the other two both real and positive, or both real and negative, or imaginary. When z_1 is positive, and the two remaining roots of (q') also real and positive, the radicals $\sqrt{z_1}$, $\sqrt{z_2}$, $\sqrt{z_3}$ are all real, and likewise the four roots of (q). If z_1 is positive, and the two remaining z_1, z_2, real and negative, then the radical $\sqrt{z_1}$ is real, but the two $\sqrt{z_2}$, $\sqrt{z_3}$ are both imaginary; and consequently, all the roots of (q), or at least two of them, are imaginary; for when $z_3 = z_2$, $+\sqrt{z_2} - \sqrt{z_3} = 0$; and therefore, in two of the preceding values of the roots of (q) the imaginary terms must disappear. If z_1 is real, and the other two roots of (q') are imaginary, first, z_1 must be positive; because, supposing $h + k\sqrt{-1}$ to be the form of z_2, the form of z_3 (102) ought to be $h - k\sqrt{-1}$; hence, $z_2 . z_3$ ($= u^2 + v^2$) is a positive product; and, consequently, z_1, z_2, z_3 cannot be positive unless z_1 is positive. Secondly, in this case, two of the roots of (q) will be real and two imaginary; because $(h \pm k\sqrt{-1})^{\frac{1}{2}}$ is equivalent to an imaginary expression of the same form (102): for instance, $a \pm b\sqrt{-1}$; therefore, in those values of the roots of (q) in which $\sqrt{z_2}$, $\sqrt{z_3}$ are taken with the same sign, the imaginary term $b\sqrt{-1}$ disappears; hence, two of the roots of (q) are real and two imaginary.

Example. § 109. To resolve now the equation

$$x^4 - 12x^2 - 16\,.\,3^{\frac{1}{2}}x - 16 = 0,$$

compare it with the general equation (q). We will have

$$G = -12,\ H = -16\,.\,3^{\frac{1}{2}},\ K = -16.$$

Hence (q') will be

$$z^3 - 6z^2 + 13z - 12 = 0,$$

whose roots are

$$z_1 = 3,\ z_2 = \frac{3 + 7^{\frac{1}{2}}\sqrt{-1}}{2},\ z_3 = \frac{3 - 7^{\frac{1}{2}}\sqrt{-1}}{2}.$$

Now, $$\left(\frac{7^{\frac{1}{2}} \pm \sqrt{-1}}{2}\right)^2 = \frac{1}{4}(7 \pm 2\,.\,7^{\frac{1}{2}}\sqrt{-1} - 1),$$
$$= \frac{1}{4}(6 \pm 2\,.\,7^{\frac{1}{2}}\sqrt{-1}),$$
$$= \frac{3 \pm 7^{\frac{1}{2}}\sqrt{-1}}{2}.$$

Hence,

$$\sqrt{\frac{3 + 7^{\frac{1}{2}}\sqrt{-1}}{2}} \text{ or } \sqrt{z_2} = \sqrt{\left(\frac{7^{\frac{1}{2}} + \sqrt{-1}}{2}\right)^2} = \frac{7^{\frac{1}{2}} + \sqrt{-1}}{2},$$

$$\sqrt{\frac{3 - 7^{\frac{1}{2}}\sqrt{-1}}{2}} \text{ or } \sqrt{z_3} = \sqrt{\left(\frac{7^{\frac{1}{2}} - \sqrt{-1}}{2}\right)^2} = \frac{7^{\frac{1}{2}} - \sqrt{-1}}{2}.$$

Now H the last term of the given equation is negative; therefore, the formulas giving the roots are (108)—

$$x_1 = \sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3},$$
$$x_2 = \sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3},$$
$$x_3 = -\sqrt{z_1} - \sqrt{z_2} + \sqrt{z_3},$$
$$x_4 = -\sqrt{z_1} + \sqrt{z_2} - \sqrt{z_3}.$$

Hence the roots of the given equation:

$$x_1 = \sqrt{3} + \sqrt{7} \qquad x_3 = -\sqrt{3} - \sqrt{-1},$$
$$x_2 = \sqrt{3} - \sqrt{7} \qquad x_4 = -\sqrt{3} + \sqrt{-1}.$$

CHAPTER II.

RATIOS, PROPORTIONS, AND PROGRESSIONS.

Division of the chapter: definitions.

§ 110. Ratios are the elements out of which *proportions* are made, which are either *simple*, or *compound*, or *continual.*

The terms of a continual proportion form a *progression.*

Now ratios are of two different kinds—namely, *arithmetical* and *geometrical.* Hence the corresponding proportions and progressions are likewise of two different kinds, distinguished from each other by the same appellations, viz.: arithmetical and geometrical. The present chapter, therefore, may be conveniently divided into two articles; in the first of which we will treat of arithmetical, and in the second, of geometrical ratios, proportions, and progressions.

ARTICLE I.

Arithmetical Ratios, Proportions, and Progressions.

Definitions and property.

§ 111. Ratios.—The difference $a - b$ between two quantities is called also arithmetical ratio, and the first of two terms is called *antecedent,* the latter, *consequent.* Now $a - b$, which we may express also by d, is such, that if we add to both terms or take from them the same quantity, the ratio or difference d is not changed. Hence, generally,

The terms of any arithmetical ratio may be increased or diminished by any quantity q *without changing the ratio itself.*

Simple arithmetical proportions.

§ 112. PROPORTIONS.—Two or more ratios equal to one another form a proportion; for instance, $a - b = a' - b'$ is a *simple* arithmetical proportion, which is either written with the sign of equality between the ratios, or more commonly as follows:

$$a - b \therefore a' - b';$$

and we read it a is to b as a' is to b'; that is, the sign — stands for *is to*, and ∴ for *as*.

The terms a and b', the first, namely, and the last, are called *extremes*, and the two remaining *mean* terms; and since from $a - b = a' - b'$, we have

$$a + b' = a' + b,$$

so also in arithmetical proportions,

Properties.

The sum of the extremes is equal to the sum of the mean terms.

And since from any equation, like $a + b' = a' + b$, we deduce $a - b = a' - b'$, so, *vice versâ*,

Whenever the sum of two terms is equal to the sum of two other terms, the four terms are arithmetically proportional.

When the mean terms of the proportion are equal to each other, the proportion becomes

$$a - b = b - b',$$

from which $2b = a + b'$, and

$$b = \frac{a + b'}{2}.$$

The term b is called *mean arithmetical proportional* between a and b', but b is given by $\frac{a+b'}{2}$; hence, to find out the mean arithmetical proportional between two given terms m and n, it is enough to divide their sum by 2.

Continual and compound proportions.

The proportions having the mean terms equal to one another, are called also continual proportions.

Let now different arithmetical proportions be given, as follows:

$$a-b=a'-b',\ c-d=c'-d',\ e-f=e'-f'\ldots.$$

It is easy to see that we will have also

$$(a+c+e+\ldots.)-(b+d+f+\ldots.)=(a'+c'+e'+\ldots) \\ -(b'+d'+f'+\ldots),$$

which is a compound proportion of those given. The ratios also, for the same reason, are called *compound* ratios.

Terms of an unlimited progression.

§ 113. PROGRESSIONS.—A progression, as we have said already, is represented by the terms of a continual proportion.

Let now a continual proportion, containing an unlimited number of ratios, be given as follows:

$$a-b=b-b'=b'-b''=b''-b'''=\&c.;$$

in this case $a,\ b,\ b',\ b'',\ b''',\ \&c.,$

are the terms of an unlimited arithmetical progression. But the general formula of any such progression may be differently expressed. For since the difference is the same for every one of the ratios $a-b$, $b-b'\ldots.$, the binomials also $b-a$, $b'-b\ldots.$ must all give the same difference. Calling δ this last difference, we will have

$$b-a=\delta,\ b'-b=\delta,\ b''-b'=\delta\ldots..$$

But from these equations we have

$$\begin{aligned} b &= a+\delta, \\ b' &= b+\delta = a+2\delta, \\ b'' &= b'+\delta = a+3\delta,\ \&c. \end{aligned}$$

Hence the terms of any arithmetical progression, from the first to the n^{th}, may be generally expressed as follows:

General formula.

$$\left.\begin{array}{l} a,\ a+\delta,\ a.+2\delta,\ a+3\delta,\ldots\ldots \\ a+(n-2)\delta,\ a+(n-1)\delta \end{array}\right\}(t).$$

With such a form given to the terms of the geometrical progression, it is easy to obtain the sum of any number of its terms, commencing with the first; for instance, the sum of all the n terms as above. Observe, in fact, that the sum of the first and second terms, is

$$2a+(n-1)\delta;$$

but the same is the sum of the terms $a+\delta$ and $a+(n-2)\delta$, that is of the second term, and of the term before the last; and the same is that of the third term, and of the next term before the last, and so on.

Suppose now, that the number n of the terms is an even number, we will evidently have $\frac{n}{2}$ sums, and each one of them equal to $2a+(n-1)\delta$; and therefore, the sum of the sum of the whole progression will be $\frac{n}{2}[2a+(n-1)\delta]$.

But let n be an uneven number, then in the progression there will be a central term, having $\frac{n-1}{2}$ terms before, and $\frac{n-1}{2}$ terms after it. These terms, added respectively to one another, as above, will give $\frac{n-1}{2}$ sums, each equal to $2a+(n-1)\delta$; and consequently, the sum of the n terms of the progression, with the exception of the central one, is $\frac{n-1}{2}[2a+(n-1)\delta]$. But the central term, added to itself, must give the sum $2a+(n-1)\delta$, as the equidistant terms do when added to one another; therefore, the central term is equal to $\frac{1}{2}[2a + (n-1)\delta]$. Hence, the sum of all the n terms of the progression, is

$$\frac{n-1}{2}[2a + (n-1)\delta] + \frac{1}{2}[2a + (n-1)\delta];$$

that is,

$$\frac{n}{2}[2a + (n-1)\delta],$$

expressed, namely, in the same manner as when n is even. Hence, generally, whatever be the number n of the terms of the progression, their sum is given by the formula,

Sum.

$$S = \frac{n}{2}[2a + (n-1)\delta].$$

That is, to know what the sum of n terms of the arithmetical progression is, it is enough to know the first term a, and the difference δ between two successive terms: for $2a + (n-1)\delta$, multiplied by one-half the given number n, gives for product the required sum.

Examples. Let us see an example. Suppose a clock striking the hours and the quarters in this manner: The hours alone, and the quarters also alone; first one, then two, and lastly three. Hence, 7 will be the number of the strokes, from the first hour, or hour one to two, and 8 will be the number of strokes from two to three, and then 9, and so on. How many strokes are contained in 12 hours? The number n of the terms is 12, the first term a is 7, and the difference δ between two successive terms is 1; therefore, the sum of the number of strokes in twelve hours is

$$S = 6(14 + 11) = 150.$$

But suppose that the hour is repeated each time when the clock strikes the quarters, and that it strikes four quarters before each hour. From the hour one to two, including the four quarters before the hour, we will have 14 strokes; from two to three, 18; and then 22, and so on. If, therefore, we ask the sum of the strokes in 12 hours, it will be given by

$$S = 6[2.14 + 11.4] = 432.$$

How and when the terms of the progression may be found.

When the first and last terms of an arithmetical progression are given, and their number is also given, we may find all the intervening terms. For let a be the first given term, and u the last, and let n be the given number of terms. The form and value of the last term u is from the preceding (t),

$$u = a + (n - 1)\delta,$$

in which equation u and a and n are known, and consequently δ is easily found: and since from the same (t) $a + \delta$, $a + 2\delta$ .... are the intervening terms between the first and the last, they also are all equally determined.

Example.

Let, for instance, the given values be as follows:

$$a = 2,\ u = 14,\ n = 5;$$

from $u = a + (n - 1)\delta$, we will have

$$14 = 2 + 4.\delta;$$

and consequently, $\delta = 3$;

hence, for the intervening terms between a and u, we have

5, 8, 11.

ARTICLE II.

Geometrical Ratios, Proportions, and Progressions.

Definitions and property.

§ 114. RATIOS.—The quotient $\frac{a}{b}$ of two quantities a and b is called also their geometrical ratio, and a the *antecedent*, and b the *consequent* of the ratio.

Whenever a ratio is mentioned without adding the quality of arithmetical or geometrical, it is always understood to be a geometrical ratio.

Multiplying now both terms of $\frac{a}{b}$ by q, we will have

$$\frac{a.q}{b.q}=\frac{a}{b}.\frac{q}{q}=\frac{a}{b};$$

and dividing the same terms by q, we have

$$\frac{a}{q}.\frac{b}{q}=\frac{a}{b}.\frac{q}{q}=\frac{a}{b};$$

that is, *The terms* a *and* b *of the ratio* $\frac{\mathrm{a}}{\mathrm{b}}$ *can be multiplied or divided by the same quantity* q *without changing the ratio.*

Variable ratios —direct and reciprocal terms.

§ 115. The terms of a ratio may be either constant or variable, and when they are variable they may vary with a certain dependence on one another, or not. If they vary independently of one another, the ratio itself is variable. But with changeable terms, depending on one another, the ratio may be constant. Suppose, for example,

$$y=\frac{1}{m}x,$$

and in this equation m to be constant. It is evident first, that for any change of x, a corresponding change must be made in y. Call y' the value to be given to y when x is changed into x', and y'' the value to be given to y when x is changed into x'', and so on. Now, generally, whatever be the values of x and y, from the given equation we always have

$$\frac{x}{y}=m.$$

Although, therefore, the terms x and y are variable, their ratio m is constant, and $\frac{x'}{y'}$, $\frac{x''}{y''}$, ... will be all equal to m. Now,

Whenever two variable quantities are so connected together as to give constantly the same ratio, they are said to vary together directly.

But taking $$\frac{1}{y}=\frac{1}{m}x$$

in the same supposition of m being constant, and x variable together with y, whatever be x and y, we will always have

$$x : \frac{1}{y} = m;$$

that is, $$x . y = m.$$

The terms x and y, therefore, vary in this case, in such a manner as to give constantly the same product. It is then plain, that one of them cannot increase without a corresponding diminution in the other, and *vice versâ*. Hence, generally,

When two variable quantities are depending on each other in such a manner as to give constantly the same product, they are said to vary inversely or reciprocally.

It is to be observed here also, that since

$$x . y = x : \frac{1}{y} = m,$$

when the variables x *and* y *vary inversely,* x *and* $\frac{1}{y}$ *vary directly.*

Continual geometrical ratios.

§ 116. We have seen (56) that irrational numbers are those limits to which an indefinite series of rational numbers of fractional form may constantly approach. So, for instance, the square roots of 2 and 3 are such numbers contained between 1 and 2, which cannot be exactly determined, but to which an indefinite series of rational numbers contained between the same limits may constantly approach.

Now all the numbers, both rational and irrational, contained within the limits 1 and 2 form a continual series; and if we conceive the number 1 to be successively changed into every one of the terms of this series, proceeding orderly from the first to the last, the number 1 would be said to increase *continually*, or to increase by degrees smaller than any assignable quantity.

Upon this, let x and y be two quantities depending on each other in such a manner that when x becomes $2x$ or $3x$, &c., y also becomes $2y$, $3y$, &c.; and when x becomes $\frac{x}{2}$, $\frac{x}{3}$, &c., y also becomes $\frac{y}{2}$, $\frac{y}{3}$, . . With regard to these variables x and y, we say that

Proposition and its demonstration. If x changes constantly, y also will change, continually keeping pace with x.

Let, in fact, m and m' represent any two whole numbers: we will have y changed into $m'y$, when x is changed into $m'x$; and if in $m'x$, we change x into $\frac{x}{m}$, in $m'y$ the variable y must become $\frac{y}{m}$. Therefore, when x is changed into $\frac{m'}{m}x$, y is necessarily changed into $\frac{m'}{m}y$. Now, m and m' are any two whole numbers; hence, $\frac{m'}{m}$ stands to represent any rational fraction; hence, also $\frac{m'}{m}$ may be any of the terms of an indefinite series, approaching constantly to some irrational number μ, and consequently, if x is changed into μx, y also will become μy.

Corollary. Generally, representing by v and v', any two numbers, either rational or irrational, when x is changed into $x' = vx$, y will become $y' = vy$, and when x is changed into $x'' = v'x$, y will become $y'' = v'y$. Hence,

$$\frac{x'}{x''} = \frac{y'}{y''}.$$

That is, when x and y change together and equally, the ratio between any two values is always equal to the ratio between the corresponding values of y.

Direct and reciprocal compound ratios.—Theorem. § 117. Let now $y, z, u, v \ldots$ represent any number of variables, all independent of one another, and let x be another variable, depending directly on each one of them, so that, for any value given to the independent variables y, $z \ldots.$ we always have the ratios $\frac{x}{y}, \frac{x}{z}, \ldots.$ unchanged.

Call now P the product $y \,.\, z \,.\, u \,.\, v \ldots.$ of the independent variables; this product depends on each one of the variables $y, z, u \ldots$ directly, and in the same manner as x. Whatever be, therefore, the values given to the independent variables, the ratio $\frac{x}{\mathrm{P}}$ will remain constantly unchanged. That is,

If x *varies directly as each one of the independent variables* y, z, u, . . . *it varies also directly as their product.*

We may arrive at the same conclusion in another manner. Since x varies directly as any of the variables $y, z, \ldots$ independent of one another, if in the ratio

$$\frac{x}{y \,.\, z \,.\, u \ldots} = \mathrm{R},$$

we give any value to y, the ratio will constantly retain the same value as R. In fact, the same ratio can be decomposed as follows:

$$\frac{x}{y}\cdot\frac{1}{z\,.\,u\,.\,.\,.\,.}=\mathrm{R}.$$

Now giving at pleasure to y any value, z, u .... suffer no change, and x varies directly as y; therefore both factors $\frac{x}{y}$ and $\frac{1}{z\,.\,u\,.\,.\,.}$ remain unchanged for any value whatever of y; that is, the ratio $\frac{x}{y\,.\,z\,.\,u\ldots}$ will be constantly equal to the same R, whatever be the value we give to y. But the same reasoning is applicable to z, to u, &c.; therefore, whatever be the values given to the independent variables y, z, u ... in the ratio $\frac{x}{y\,.\,z\,.\,u\,.\,.\,.}$, its value will remain constantly unchanged; that is, x varies directly as the product $y\,.\,z\,.\,u$ ....

Corollary.

But if x depends directly on u, v, ... and inversely on y, z, ... x varies directly (115) as each of the following:

$$u,\ v,\ \ldots\ \frac{1}{y},\ \frac{1}{z}\ \ldots$$

Therefore, it varies directly as their product $u\,.\,v\ldots\frac{1}{y}\,.\,\frac{1}{z}\ldots$ $=\frac{u\,.\,v\,.\,.\,.}{y\,.\,z\,.\,.\,.}$; hence,

Theorem.

If x *varies directly as* u, v, ... *and inversely as* y, z, ... *it will vary directly as the quotient* $\frac{u\,.\,v\,.\,.\,.\,.}{y\,.\,z\,.\,.\,.}$.

Simple geometrical proportions.

§ 118. PROPORTIONS.—Two or more ratios equal to each other, form a proportion; for instance,

$$\frac{a}{b}=\frac{a'}{b'},$$

and this is the general expression of any simple geometrical proportion. The manner, however, of writing these proportions is as follows:

$$a:b::a':b',$$

and we read it a is to b as a' is to b'; that is, the two dots [:] stand for *is to*, and the four dots [: :] stand for *as*. In geometrical proportions also, the terms a and b' are called extremes, and the other two mean terms.

General properties.

From the proportion equivalently represented by the equation

$$\frac{a}{b}=\frac{a'}{b'},$$

we have $\frac{a}{b}\cdot\frac{b'}{b'}=\frac{a'}{b'}\cdot\frac{b}{b};$

and consequently $a\,.\,b'=a'\,.\,b.$

That is, in geometrical proportions,

The product of the extremes is equal to that of the mean terms.

But from $a\,.\,b'=a'\,.\,b$ we have likewise, $\frac{a}{b}=\frac{a'}{b'}$. Hence,

When four terms a, b, a′, b′ *are such that the product of the first by the last is equal to the product of the other two terms, the four terms are geometrically proportional.*

Continual and compound proportions.

Suppose now that the mean terms are equal to each other, then we will have

$$a : b :: b : b';$$

and consequently, $b^2=ab',$

that is, $b=\sqrt{ab'}.$

b is the *mean geometrical proportional* between a and b'. If, therefore, a and b' are given, to find out their mean geometrical proportional, take the square root of their product.

Such proportions also, having both mean terms equal, are called *continual* proportions.

Let now several proportions be given, as follows

$$\frac{a}{b}=\frac{a'}{b'},\ \frac{c}{d}=\frac{c'}{d'},\ \frac{e}{f}=\frac{e'}{f'}\cdots\cdot$$

We have from them

$$\frac{a}{b}\cdot\frac{c}{d}\cdot\frac{e}{f}\cdots=\frac{a'}{b'}\cdot\frac{c'}{d'}\cdot\frac{e'}{f'}\cdots\cdot$$

And consequently

$$(a.c.e\ldots) : (b.d.f\ldots) :: (a'.c'.e'\ldots) : (b'.d'.f'\ldots)$$

That is, *If the first terms of several or any number of geometrical proportions be multiplied together, and likewise the second, third, and fourth terms, the products are proportional.*

The ratios and the proportion itself, made out of these products, are called *compound ratios* and *compound proportion*.

Other properties of geometrical proportions.

§ 119. From $a : b :: a' : b'$, or from

$$\frac{a}{b} = \frac{a'}{b'},$$

we have

$$\frac{a}{a'} = \frac{b}{b'};$$

that is, $$a : a' :: b : b'.$$

Hence the terms of any geometrical proportion are such, that

The antecedent of the first ratio is to the antecedent of the second ratio as the consequent of the first is to the consequent of the second ratio.

Again, from the given proportion or equation $\frac{a}{b} = \frac{a'}{b'}$, we have

$$\frac{b}{a} = \frac{b'}{a'};$$

that is, $$b : a :: b' : a'.$$

Hence, *The consequent of the first ratio of any given proportion is to its antecedent as the consequent of the second ratio is to its own antecedent.*

From the same proportion or equation $\frac{a}{b} = \frac{a'}{b'}$, we deduce the two following:

$$\frac{a}{b} + 1 = \frac{a'}{b'} + 1, \frac{a}{b} - 1 = \frac{a'}{b'} - 1;$$

and consequently,

$$\frac{a+b}{b} = \frac{a'+b'}{b'}, \frac{a-b}{b} = \frac{a'-b'}{b'};$$

that is,

$$a + b : b :: a' + b' : b',$$
$$a - b : b :: a' - b' : b'.$$

In equal manner, since $\frac{a}{b} = \frac{a'}{b'}$ may be changed into $\frac{b}{a} = \frac{b'}{a'}$, we have

$$b + a : a :: b' + a' : a',$$
$$b - a : a :: b' - a' : a'.$$

That is, the terms of any given geometrical proportions are such, that

The sum or the difference of the terms of the first ratio is to the first or to the second term of the same ratio, as the sum or the difference of the terms of the second ratio is to the first or to the second term of the same ratio.

We may observe that in the last proportion $b - a : a :: b' - a' : a'$, the differences or terms $b - a$, $b' - a'$, may be changed into $a - b$ and $a' - b'$, the terms remaining still in geometrical proportion; for this inversion affects only the sign of the ratios, which being equally changed in both of them, the equality of the ratios still exists, and consequently the proportion also. The same observation may be made with regard to the proportion $a - b : b :: a' - b' : b'$. So that the last inference is altogether general.

From the proportions or equations

$$\frac{a+b}{b} = \frac{a'+b'}{b'}, \quad \frac{a-b}{b} = \frac{a'-b'}{b'},$$

just inferred from the given proportion or equation $\frac{a}{b} = \frac{a'}{b'}$, we have

$$\frac{a+b}{a'+b'} = \frac{b}{b'}, \quad \frac{a-b}{a'-b'} = \frac{b}{b'};$$

hence, also,

$$\frac{a+b}{a'+b'} = \frac{a-b}{a'-b'},$$

and

$$\frac{a+b}{a-b} = \frac{a'+b'}{a'-b'};$$

consequently,

$$a + b : a' + b' :: a - b : a' - b',$$
$$a + b : a - b :: a' + b' : a' - b'.$$

That is, *The sum of the terms of the first ratio of any propor-*

tion is to the sum of the terms of the second ratio, as the difference of the terms of the first ratio is to the difference of the terms of the second. And

The sum of the terms of the first ratio is to their difference, as the sum of the terms of the second ratio is to their difference.

These, and the preceding inferences, are of great practical use.

Numerical proportions whose first ratio is irreducible.

§ 120. Let the terms of the proportion

$$a : b :: a' : b'$$

be whole numbers, and let the terms a, b of the first ratio be prime numbers to each other. Of the numbers a', b', the first will be equal to na, the second to nb, n being a whole number.

In fact, calling n the quotient $\frac{a'}{a}$, or making $\frac{a'}{a} = n$, a' may always be expressed by na. But na cannot express a', unless nb expresses b'; for by supposition, $\frac{a'}{b'} = \frac{a}{b}$;

and consequently, if $a' = na$, b' cannot be but equal to nb.

We say now, that n is a whole number; for from the same equation, $\frac{a'}{b'} = \frac{a}{b}$, we have

$$a' = \frac{a}{b} . b' = \frac{ab'}{b} = a . \frac{b'}{b}.$$

Now $\frac{a}{b}$ by supposition is an irreducible fraction; therefore, $\frac{ab'}{b}$ cannot be equal to the whole number a', unless (53) the number b' is exactly divisible by b; that is, unless the quotient $\frac{b'}{b}$ is a whole number; but $b' = nb$, and from this equation, we have $\frac{b'}{b} = n$; the number n, therefore, is a whole number.

Terms of any geometrical progression.

§ 121. PROGRESSIONS.—The terms of any geometrical progression are the same as those of any continual proportion. And a continual proportion is generally represented by

$$\frac{a}{b} = \frac{b}{b'} = \frac{b'}{b''} = \ldots.$$

or else by $a : b :: b : b' :: b' : b'' :: \ldots.$

The terms, therefore, $a, b, b', b'', \ldots.$

are the terms of any unlimited geometrical progression.

Let us now call k the common value of the ratios $\frac{a}{b}, \frac{b}{b'}, \ldots$ we will have
$$\frac{a}{b}=k,\ \frac{b}{b'}=k,\ \frac{b'}{b''}=k,\ \&c.;$$
and consequently,
$$b=\frac{a}{k},\ b'=\frac{b}{k}=\frac{a}{k}:k=\frac{a}{k^2},$$
$$b''=\frac{b'}{k}=\frac{a}{k^2}:k=\frac{a}{k^3}\ \ldots\ldots;$$
hence, the terms of any geometrical progression, from the first to the n^{th}, may be generally represented as follows:
$$a,\ \frac{a}{k},\ \frac{a}{k^2},\ \ldots\ldots\ \frac{a}{k^{n-1}}\quad(\tau);$$

General formula.

or else, (making $\frac{1}{k}=z$), the general formula of the terms of any geometrical progression containing n terms, is
$$a,\ az,\ az^2,\ az^3,\ \ldots\ldots\ az^{n-1}\quad(\tau').$$
Now the sum of these n terms is easily obtained from the known product (63)
$$(1+z+z^2+\ \ldots\ldots\ +z^{n-1})(1-z)=1-z^n,$$
which gives $1+z+z^2+\ \ldots\ +z^{n-1}=\frac{1-z^n}{1-z}$,
from which
$$a+az+az^2+\ \ldots\ +az^{n-1}=a\frac{1-z^n}{1-z}=\frac{a}{1-z}-\frac{az^n}{1-z}.$$

Sum of n terms, and sum of an indefinite number of terms.

Now, the first number of this equation is the sum S of the terms (τ'); therefore,
$$S=\frac{a}{1-z}-\frac{az^n}{1-z}\quad(\sigma),$$
Suppose now that the numerical value of z is a number less than unity, it may be represented by a fraction $\frac{r}{m}$, in which m is greater than r. In this supposition, the last term of (σ) will be
$$a\left(\frac{r}{m}\right)^n:\frac{m-r}{m}=\frac{m.a}{m-r}\left(\frac{r}{m}\right)^n.$$

Now, $\frac{m.a}{m-r}$ is a constant coefficient, as well as the fraction $\frac{r}{m}$, but the exponent n has different values according as the terms summed up are more or less in number. Now the more we increase n, the more the power $\left(\frac{r}{m}\right)^n$ approaches to zero, and with it the whole term $\frac{m.a}{m-r}\left(\frac{r}{m}\right)^n$. Hence, taking an indefinite number of terms—that is, supposing the number of the terms summed up to be without limit—the last term of (σ) must disappear, and in this case,

$$S = \frac{a}{1-z} \ldots . \; (\sigma')$$

is the sum of an indefinite number of terms (τ').

Examples. It is related that the inventor of the game of chess, solicited to ask a reward, answered: Put one grain of wheat in the first square of the board, two in the second, four in the third, eight in the next, and so on, till the sixty-fourth, which is the last. How many grains of wheat did he ask?

Here we have the geometrical progression, whose terms are

$$1, 2, 4, 8, 16, \ldots\ldots$$

which, compared with the terms (τ'), give

$$a = 1, \; z = 2, \; n = 64.$$

Hence, the sum of all these terms is

$$S = \frac{1}{-1} - \frac{2^{64}}{-1}$$
$$= 2^{64} - 1.$$

In the following chapter, we will see how the power 2^{64} may be obtained and expressed by an equivalent common number.

But let, with $a = 1$, z be less than unity and equal to $\frac{1}{2}$; the terms of the progression in this case will be (τ')

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

and the sum of the same terms indefinitely protracted is (σ')

$$S = \frac{1}{1 - \frac{1}{2}} = 1 : \frac{2 - 1}{2} = 2.$$

Hence, $$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2.$$

How and when the terms of a certain progression may be found.

The first and the n^{th} terms of a certain progression being given, all the other terms may be found.

Calling a the first and u the n^{th} terms given, we have (τ') $u = az^{n-1}$, in which equation u, a, n are known elements, and consequently z may be found. And when z is obtained, all the terms a, az, az^2 .... are likewise obtained.

Example.

Let, for example, the first and fifth terms of a certain progression be as follows:

$$a = 2, \; u = 32,$$

from $u = az^{n-1}$, we have, generally,

$$z^{n-1} = \frac{u}{a};$$

that is, $$z = \sqrt[n-1]{\frac{u}{a}},$$

and in our case, since $n = 5$,

$$z = \sqrt[4]{16} = 2;$$

and, therefore, the terms of the progression are as follows:

2, 4, 8, 16 .....

CHAPTER III.

LOGARITHMS.

Exponential quantities.

§ 122. WHEN the exponent of any quantity is variable, the power is called an *exponential* quantity; thus, for instance, the power

$$a^x,$$

in which x is supposed to be a variable number, is an exponential quantity. Now a may be either variable or constant. Let a be a constant number; if instead of x we take x', x'', x'''...., we will have different powers which we may represent by the numbers z', z'', z'''.... That is,

$$a^{x'} = z',$$
$$a^{x''} = z'',$$
$$a^{x'''} = z''', \text{ \&c.}$$

Logarithms.

And we may evidently conceive an indefinite series of such numbers z', z''... depending on the variable exponent x. Now this variable exponent is called *the logarithm* of the power; that is, x' is the logarithm of z', x'' is the logarithm of z'', and so on.

But the powers depend also on the constant number a; for, supposing the same exponents x', x''... applied to a different constant, for example, to A, all the powers z', z''... will be changed into others, which we may call Z', Z''....; that is, we will have

$$A^{x'} = Z',$$
$$A^{x''} = Z'',$$
$$A^{x'''} = Z''', \text{ \&c.}$$

Here, also, x' is the logarithm of Z', x'' the logarithm of Z''.... But the change of the constant a into A, changing the whole system of numbers to which the same logarithms belong, it is plain, that when a logarithm x' or x'' is given, and the corresponding number is required, we must first know what is the constant or root to which the logarithm is applied as an exponent.

Base.

This constant is called the *Base* of the logarithms. And in the preceding examples, x', x''... are logarithms of the numbers z', z''... in the system having a for base, and logarithms of the numbers Z', Z''... in the system having A for base. We may remark also the propriety of the appellation, since the base is like a foundation on which the whole system is built.

Sign.

The sign or mark with which logarithms are indicated, is either the initial *log.* of the word logarithm, or the simple letter *l.* And when logarithms belong to different systems, to distinguish the logarithms of one system from those of another, we may put an accent to the letter *l*, or change the small *l* into a capital one. For instance, *log.* or *l.* being the sign of the logarithms in the system having a for base, we may express by *Log.* or *L.*, the logarithms of the system whose base is A. And from the preceding equations we will have for the first system,

$$x' = log.\, z',\ x'' = log.\, z'',\ \&c. \ldots$$

or,

$$x' = l.\, z',\ x'' = l.\, z'',\ \&c. \ldots$$

And for the second,

$$x' = Log.\, Z',\ x'' = L.\, Z'',\ \&c. \ldots$$

or,

$$x' = L.\, Z',\ x'' = L.\, Z'',\ \&c. \ldots$$

The logarithm of unity, and the logarithm of the base, are the same in all systems.

§ 123. It is well known (15) that any number raised to the exponent zero, gives 1 for its power; hence, $a^0 = A^0 = \ldots = 1$; that is, 0 *is the logarithm of unity in all systems.* Hence,

$$0 = l.\,(1) = L.\,(1) = \ldots\ldots$$

It is also equally known that any number raised to the exponent 1, gives the power equal to the number, that is, $a^1 = a$,

$A^1 = A$; .... hence, *Unity is the logarithm of the base in all systems.* Thus we have,

$$1 = l.\ a = L.\ A = \ldots.$$

Positive and negative logarithms.

We have seen (45) that a^{-m} is the same as $\frac{1}{a^m}$.

Theorems.

Suppose, now, a to be greater than unity; since $a^0 = 1$ and $a^1 = a$, the same quantity a raised to any positive exponent either between 0 and 1, or greater than 1, will give always a power greater than unity. Hence, when the base is greater than unity and the logarithm is positive, the number is likewise greater than unity.

But if with $a > 1$ we take a^{-1} instead of a^1, then we will have $a^{-1} = \frac{1}{a^1} = \frac{1}{a}$. Now $\frac{1}{a}$ is less than unity; hence, the number or power corresponding to a^{-1} is smaller than unity; the same is to be said of any other power a^{-m} in which m is taken between 0 and 1, or greater than 1. That is, when the base is greater than 1, and the logarithm is negative, the corresponding number is less than unity. From this and the preceding inference we deduce the two following:

When the base is greater than unity, and the number is likewise greater than unity, the logarithm of this number is positive.

When the base is greater than unity, and the number less than unity, the logarithm of the number is negative.

But let the base a be smaller than unity; then giving to a any exponent contained between 0 and 1, the resulting power will be contained between $a^0 = 1$ and $a^1 = a$. Now all the numbers between 1 and a are smaller than unity; a, therefore, raised to any exponent between 0 and 1, gives a fraction for power. But if a is raised to any exponent greater than unity, the power also will be a fraction. That is, when the base is less than unity, and the logarithm is positive, the corresponding number also is less than unity.

Give now to $a < 1$ a negative exponent, we will have generally $a^{-m} = \frac{1}{a^m}$; but a^m, as we have just seen, is always less than unity; therefore, $\frac{1}{a^m}$ is always greater than unity; hence, when the base is less than unity, and the logarithm is negative, the corresponding number is greater than unity. From this and the preceding deduction, we infer also that

When the base is less than unity, and the number also is less than unity, the logarithm of the number is positive.

When the base is less than unity, and the number greater than unity, the logarithm is negative.

In any system, when the logarithms form an arithmetical progression, the corresponding numbers form a geometrical one.

§ 124. Take now, successively for x, the equation $a^x = z$, x, $x + \delta$, $x + 2\delta$, ... which are the terms of any arithmetical progression, we will have

$$\begin{aligned} a^x &= z, \\ a^{(x+\delta)} &= z', \\ a^{(x+2\delta)} &= z'', \\ a^{(x+3\delta)} &= z''', \text{ \&c.} \end{aligned}$$

Now (42) $a^{x+\delta} = a^x . a^\delta$, $a^{x+2\delta} = a^x . a^{2\delta} = a^x(a^\delta)^2$, $a^{x+3\delta} = a^x(a^\delta)^3$, ... and making $a^\delta = \zeta$, we will have

$$a^{x+\delta} = a^x\zeta, \quad a^{x+2\delta} = a^x\zeta^2, \quad a^{x+3\delta} = a^x\zeta^3, \text{ \&c.}$$

Hence, the powers z, z', z'', z''' .... are represented by the terms a^x, $a^x\zeta$, $a^x\zeta^2$, $a^x\zeta^3$, ... which are the terms of a geometrical progression. But the same powers are the numbers corresponding to the logarithms x, $x + \delta$, $x + 2\delta$ ... in the system having for base any number a. Therefore,

In any system of logarithms, when the logarithms form an arithmetical, the corresponding numbers form a geometrical, progression.

Useful theorems.

§ 125. But let us come to those theorems which show how advantageously logarithms may be used.

Let x and y be the logarithms of the numbers z and v in the system having a for base, we will have

$$a^x = z,\ a^y = v;$$

and consequently, $a^x . a^y = a^{x+y} = z . v.$

Now, from these equations, we have, also

$$x = l.z,\ y = l.v,\ x + y = l.(z . v),$$

and therefore, $l.(z . v) = l.z + l.v;$

Theorem 1. that is to say, *The logarithm of the product is equal to the sum of the logarithms of the factors.*

Again, from the same equations, we have

$$\frac{a^x}{a^y} = \frac{z}{v},\ \text{or}\ a^{x-y} = \frac{z}{v};$$

and consequently, $x - y = l.\frac{z}{v},$

or, $l.\frac{z}{v} = l.z - l.v;$

Theorem 2. that is, *The logarithm of the quotient is equal to the logarithm of the numerator, minus the logarithm of the denominator.*

Raise to the exponent c both members of the equation

$$a^x = z,$$

we will have $(a^x)^c = z^c$

or $a^{xc} = z^c,$

and $l.z^c = xc;$

but from $a^x = z$, we have $x = l.z$; hence,

$$l.z^c = cl.z;$$

Theorem 3. that is, *The logarithm of the power of any number is equal to the logarithm of the number multiplied by the exponent.*

But if we take the root of the degree c of both members of the equations $a^x = z$, we will have

$$a^{\frac{x}{c}} = z^{\frac{1}{c}};$$

and consequently, $l.z^{\frac{1}{c}} = \frac{x}{c} = \frac{1}{c}x;$

now $x = l.z$; hence,

$$l.z^{\frac{1}{c}} \text{ or } l.\sqrt[c]{z} = \frac{1}{c}l.z;$$

Theorem 4. that is, *The logarithm of the root of any number is equal to the logarithm of the number divided by the degree or index of the root.*

From these theorems we infer, that when the logarithms of the numbers are determined in any system, numerical calculations become much easier; for multiplications and divisions are performed with simple additions and subtractions—powers and roots are obtained with multiplications and divisions.

Common or ordinary tables of logarithms. The logarithms of numbers have been carefully determined, and the common and most useful system of logarithms is that whose base is $a = 10$; hence, the general formula $a^x = z$, in this system becomes

$$10^x = z,$$

and taking in it successively 0, 1, 2 .... instead of x, we will have 1, 10, 100 .... for the corresponding number z.

To find out the logarithms of the intervening numbers between 1 and 10, between 10 and 100, &c., it will be enough to take the numbers between these limits in a geometrical progression, and an equal number of terms between 0 and 1, between 1 and 2, &c., in an arithmetical progression; the terms of the latter progression will be respectively logarithms (124) of the corresponding terms of the geometrical progression.

Now the terms of any progression, either arithmetical or geometrical, are the same (113, 121) as those of a continual proportion; and when the extreme terms a and b' of a continual proportion are given, we obtain the mean arithmetical term by taking (112) $\frac{a+b'}{2}$, and the mean geometrical by taking (118) $\sqrt{ab'}$. Hence, in our case, the mean geometrical term between 1 and 10 is $\sqrt{10}$ (= 3, 16 ...), and the mean arithmetical term between 0 and 1 is $\frac{0+1}{2} = \frac{1}{2}$, and the numbers, 1, 3, 16 10,

$$0, \quad \frac{1}{2}, \quad 1,$$

are terms of two progressions, the first geometrical and the second arithmetical; and consequently, since 0 and 1 are respectively logarithms of 1 and 10, so $\frac{1}{2}$, or 0.5, is the logarithm of 3, 16 ... Now, again, taking the mean geometrical proportional between 1 and 3, 16 ... and between 3, 16 .... and 10; taking, also, the mean arithmetical proportional between 0 and 0.5, and between 0.5 and 1, we will have two more numbers contained between 1 and 10 and their logarithms; continuing in this manner, we may have as many numbers as we like between 1 and 10 and their logarithms. The same may be said of the numbers contained within the limits 10 and 100, 100 and 1000, &c., and of their logarithms. This method shows well how logarithms of any quantity of numbers may be found; in practice, however, methods more expeditious are preferred. It is yet to be remarked that even the method just explained is not necessarily to be applied to all the numbers; but it is enough to find the logarithms of prime numbers, for these being determined, we have the logarithms also of all the numbers which can be resolved into factors, and the logarithms of fractions also. Take the number 15, for example, which may be decomposed into the two prime numbers and factors 3.5, we will have (125, Th. 1) $l.15 = l.3 + l.5$; take the fraction $\frac{7}{9}$, we will have (125, Th. 2) $l.\frac{7}{9} = l.7 - l.9$. But it is not necessary for us to dwell any longer on this subject, for copious tables of logarithms are made with most exquisite accuracy and with all desirable improvements.

Constant ratio of logarithms in every system.

§ 126. Let a, a', a'' represent the bases of three different systems of logarithms, and $l.$, $l.'$, $l.''$ the signs of the corresponding logarithms. From the equation $$a'^x = r,$$ we have, $$x = l.'r.$$ But if we take the logarithm of each member of the same equation in the system having a for base, and then in the system having a'' for base, we will have,

$$l.(a'^x) = l.r, \; l.''(a'^x) = l.''r,$$

or, (125, Th. 3,)

$$xl.a' = l.r, \; xl.''a' = l.''r.$$

Substituting now in these equations the preceding value of x, we have, $$l.'r.l.a' = l.r, \; l.'r.l.''a' = l.''r.$$

Hence, $$\frac{l.r}{l.a'} = l.'r, \; \frac{l.''r}{l.''a'} = l.'r.$$

And consequently,

$$\frac{l.r}{l.a'} = \frac{l.''r}{l.''a'} = \ldots\ldots = l.'r;$$

that is to say,

The logarithms of any two numbers r, a', *divided by each other, give constantly the same ratio in all systems.*

How the logarithm of one system may be inferred from the logarithms of others.

Suppose the logarithms $l.$ of the system having a for base, to be known or determined, and let a' be the base of another system of logarithms $l.'$ which are to be determined; let also n be any number. The logarithm of n is known in the system having a for base, and unknown in the system whose base is a'; that is, in the equation $$a'^x = n,$$ the exponent $x (= l.'n)$ is unknown. But from the same equation, taking the logarithms in the system whose base is a, we have $$xl.a' = l.n;$$

and consequently, $$x = \frac{l.n}{l.a'};$$

but the logarithms $l.$ of n and a' are known; hence their ratio also, or x, is known. Knowing, therefore, the logarithms of numbers in any system, we may from them infer the logarithm of any number n taken in any other system; and consequently, when tables of logarithms are made for one system, we may derive from them other tables for any other system of logarithms.

Explanatory remarks. Rules.

§ 127. Let us resume the two progressions,

$$1,\ 10,\ 100,\ 1000 \ldots.$$
$$0,\ 1,\ 2,\ 3 \ldots.$$

the first representing the numbers, and the second the corresponding logarithms, in the system having $a = 10$ for base.

In the same system the logarithms of all fractions must be (123) negative, and the following terms may be added to the preceding progressions,

$$\ldots.\frac{1}{1000},\ \frac{1}{100},\ \frac{1}{10},$$
$$\ldots\ -3,\ -2,\ -1;$$

so that the number 1 in the geometrical, and 0 in the arithmetical, progressions, are the central or middle terms of two progressions, indefinite in both ways.

From the same progressions we see that in the same manner in which the logarithms of the numbers between 1 and 10 are greater than 0, and smaller than 1, the logarithms of the numbers between 10 and 100 are greater than 1, and smaller than 2, and so on. In like manner, the logarithm of the fractions between $\frac{1}{10}$ and 1 are contained between -1 and zero, and the logarithms of the fractions between $\frac{1}{100}$ and $\frac{1}{10}$ are contained between -2 and -1, &c.

Calling v any number between 1 and 10, since all the numbers between 10 and 100 are ten times greater than the corresponding numbers between 1 and 10, the number contained between 10 and 100 and corresponding to v, will be $10v$ or av, and so likewise the next corresponding number between 100 and 1000 will be $100v$ or a^2v, and so on In like manner, since the numbers between $\frac{1}{10}$ and 1 are ten times less than the corresponding numbers between 1 and 10, the fraction between $\frac{1}{10}$ and 1, corresponding to ν, is $\frac{\nu}{10}$ or $\frac{\nu}{a}$, and the next fraction corresponding to the same ν, and contained between $\frac{1}{100}$ and $\frac{1}{10}$, is $\frac{\nu}{100}$ or $\frac{\nu}{a^2}$, and so on.

So that, we may generally represent by $a^n\nu$ any number contained between the decades 10, 100, 1000 .... and by $\frac{\nu}{a^n}$, any number contained between the decadal fractions 1, $\frac{1}{10}$, $\frac{1}{100}$, ... giving, namely, to n any of the numerical values 1, 2, 3 ..... And to represent all the numbers, we have a^n, $a^n\nu$

for those above unity, and $\frac{1}{a^n}$, $\frac{\nu}{a^n}$

for the fractions. Whatever, therefore, may be said concerning these numbers and their logarithms is evidently applicable to all numbers and logarithms in our system. From the same formulas we infer general rules, useful both for the understanding and the use of the tables.

But first observe that the immediate object of logarithmical tables is twofold. To point out, namely, the logarithm corresponding to a given number, or, *vice versâ*, to point out the number corresponding to a given logarithm.

It is scarcely necessary to say any thing concerning the numbers a^n, $\frac{1}{a^n}$ $(= a^{-n})$ of a mere decadal form, it being evident that a^n is equal to unity followed by as many zeros as there are units in n, and $\frac{1}{a^n}$ is equal to 1 divided by unity, followed by as many zeros as there are units in n. And, *vice versâ*, when any whole number of a mere decadal form is given, its logarithm n will be a whole number containing as many units as there are zeros in the given number. Hence,

Rule 1. *When the given whole number* N *is of a mere decadal form, it has for logarithm a number containing exactly as many units as there are zeros in* N.; *and when the given logarithm* n *is an exact whole number, the corresponding number is unity followed by* n *zeros.*

It is plain that for such logarithms and numbers we need not have recourse to the tables, and so also for the fractional numbers $\frac{1}{\mathrm{N}}$ of simply decadal form, for which, and for their logarithms, we have the following rule:

Rule 2. *When the given fractional number* $\frac{1}{\mathrm{N}}$ *is of simply decadal form, it has* — n *for its logarithm, containing exactly as many units as their are zeros in* N; *and when the given logarithm* — n *contains an exact number of units, the corresponding number is* 1 *divided by unity, followed by* n *zeros.*

We may observe, that such fractions of simply decadal form may be expressed also by 0,1, 0,01, 0,001 ....; and using the decimal form instead of that of ordinary fractions, the second rule will be modified as follows:

When a decimal fraction ends with 1, *preceded by* n *ciphers, all equal to zero, the logarithm of the fraction is* — n. *And when, vice versâ,* — n *is given, the corresponding number or decimal fraction ends with* 1, *preceded by* n *ciphers, all equal to zero.*

Let us now come to the numbers $a^n\nu$, $\frac{a^n}{\nu}$, and to their logarithms, in which ν, we must recollect, is any number greater than 1, and less than 10. But $a^{o,\delta}$ also, in which o,δ represents any decimal fraction, is a number, and any number contained between 1 and 10; therefore, we may generally write $\nu = a^{o,\delta}$,

and in this equation, the exponent o,δ cannot be changed except when ν is changed. Now with $\nu = a^{o,\delta}$, we have also

$$a^n\nu = a^n \,.\, a^{o,\delta} = a^{n+o,\delta} = a^{n,\delta},$$

and whatever be n, the decimal fraction δ will always be the same when ν remains the same; but from $a^{n,\delta} = a^n\nu$, we have

$$n,\delta = l.(a^n.\,\nu) \quad (\lambda).$$

And $a^n.\,\nu$ is a number contained between a^n and a^{n+1}, either a simple whole number or a whole number with a fraction $\triangle$ added to it.

In both cases the integral part (let us call it N′) of $a^n . \nu$ must have the same number of ciphers that are in a^n; that is, in

$$a^n . \nu = \mathrm{N}', \triangle \quad (\lambda'),$$

the number of ciphers of N′ is the same as the number of those of a^n, namely, $n + 1$.

The figures, besides, of the number N′, △ are the same as the figures of ν, the first $n + 1$ of which form the integral part N′, and the other, if there are any, the decimal △. Now from (λ) and (λ'), we have $n, \delta = l.(\mathrm{N}', \triangle)$.

And from the preceding remarks, it follows first, that the figures of the integral part N′ of N′, △ are one more than the units contained in the integral part n of the logarithm, and *vice versâ*.

Secondly, the figures of N′, △ are invariably the same when the fractional part δ of the logarithm remains the same, and *vice versâ*, for the figures of N′, △ are the same as those of ν, and δ does not change except with ν.

Hence, it is enough to know what is the ν corresponding to the fraction o, δ to have immediately the numbers corresponding to all the following logarithms, $1, \delta$, $2, \delta$, $3, \delta \ldots . m, \delta$,
and *vice versâ*, when the number is given, and consequently, the ciphers of ν also are given, it is enough to know what is the fraction o, δ corresponding to ν, to find out also the logarithm of the given number.

Now this is precisely that which is given by the tables. That is, the first column, marked N, contains the numbers or figures of ν, and the other columns the decimal part o, δ of the logarithms. Hence,

To find the logarithm when the number is given, we have the following rule:

Rule 3. *When the number* N′, △ *is given, write* n *containing one unit less than the number of figures in* N′, *and this* n *is the integral part of the logarithm. Then, taking* N′, △ *as an uninterrupted number, add to* n *the fraction* o, δ, *corresponding to the same number, and given by the tables.*

Characteristic. The whole number n, or integral part of the logarithm n, δ, is called the *characteristic*. Hence, *The characteristic of the logarithm of any given number contains one unit less than the number of figures forming the integral part of the given number.*

Rule 4. *When the logarithm* n, δ *is given, find in the tables the number corresponding to* δ*; cut off the first* $\mathrm{n} + 1$ *figures of this number from the following: the first part will be the integral, and the rest the decimal part of the number having* n, δ *for its logarithm.*

With regard to the fraction $\frac{\nu}{a^n}$ we may remark first, that since a^n is a number of simply decadal form, like 10, 100, &c., the quotient $\frac{\nu}{a^n}$ reduced to the form of a decimal fraction will contain the same figures that are in ν, preceded by one or more zeros; that is, as many in number as there are units in n.

Observe, secondly, that $\frac{\nu}{a^n} = a^{-n}. \nu$; hence,

$$\frac{\nu}{a^n} = a^{-n}. a^{0,\delta}.$$

Now, $a^{-n}. a^{0,\delta} = a^{-n+0,\delta}$. Instead, however, of writing explicitly the difference $-n + 0, \delta$, the same exponent is represented by the simple expression $\bar{n}, \delta$, with the negative sign above the characteristic, to signify that it does not affect the decimal part δ added to it. We will have then,

$$\frac{\nu}{a^n} = a^{\bar{n},\delta};$$

the two numbers ν and δ depending on each other, as above, whatever n should be. But $\frac{\nu}{a^n}$, reduced to the form of a decimal fraction, may be represented by $0, \mathrm{D}$; hence,

$$a^{\bar{n},\delta} = 0, \mathrm{D};$$

that is,

$$\bar{n}, \delta = l\,(0, \mathrm{D}).$$

Now, from the tables we may have the ciphers of ν corresponding to δ, or, *vice versa*, we may have δ corresponding to the ciphers of ν that are in D; and since, as we have observed, the figures of ν commence in $0, \mathrm{D}$ after n zeros, hence, For negative logarithms and their corresponding members or fractions, we have the following rules.

First, when the fraction 0, D is given, and its logarithm is to be found:

Rule 5. *See how many zeros precede the first figure of units in* 0, D, *and write the number* $\bar{n}$ *of these zeros as the characteristic of the logarithm; taking then from* D *the number which commences with the first figure of units, find from the tables the corresponding* δ, *and add it to the characteristic.*

And to find the number corresponding to a given logarithm, we have the rule—

Rule 6. *When the logarithm* $\bar{n}$, δ *is given, write first as many zeros as there are units in* n, *separating with a comma the first from the others; then add to these ciphers the number corresponding to* δ, *as given by the tables.*

We may observe, that the logarithms of fractional numbers are differently expressed by different writers. We have expressed them by $\bar{n}$, δ. But when $n = 1$, or 2, &c., others express these logarithms by 9, δ; 8, δ, &c. But this manner of writing such logarithms is somewhat ambiguous, and we may say partial. For this reason we have preferred to make use of the above-mentioned expression.

Application of logarithms. § 128. The practical application of the preceding rules must be left entirely to the direction of the teacher, and to the diligence of the pupil; since any attempt to apply them without having logarithmical tables at hand would prove altogether useless.

To give, however, some idea of the useful application of logarithms, let us observe that exponential equations can be resolved by means of logarithms. That is, those equations in which the unknown quantity is the exponent of some other quantity; as, for instance, in

$$\frac{c}{h} = \frac{1}{q^{(x-1)}},$$

in which equation x is the unknown quantity.

Applying the logarithms, we will have

$$l.\frac{c}{h} = l.\left[\frac{1}{q^{x-1}}\right]$$

hence, (125, Th. 2, 3,)

$$l.c - l.h = l.1 - l.q^{(x-1)} = l.1 - (x-1)\,l.q;$$

and since $l.1 = 0$,

$$l.c - l.h = (x-1)\,l.q;$$

from which

$$x - 1 = \frac{l.c - l.h}{l.q},$$

and

$$x = \frac{l.c - l.h}{l.q} + 1.$$

Let another exponential equation be as follows:

$$\left(\frac{101}{100}\right)^x = 10.$$

Taking the logarithms, we will have

$$x\,l.\frac{101}{100} = l.10;$$

that is,

$$x\,[l.101 - l.100] = 1.$$

Now, from the tables $l.100 = 2$, and $l.101 = 2{,}0043214$; hence, $l.101 - l.100 = 0{,}0043214$; therefore,

$$x = \frac{1}{0{,}0043214} = \frac{1{,}0000000}{0{,}0043214};$$

and finally the approximate value of

$$x = 231.$$

But suppose that instead of $\frac{101}{100}$, the given fraction is $f = \frac{100}{101}$; then, from the equation

$$\left(\frac{100}{101}\right)^x = 10,$$

we have

$$x\,[l.100 - l.101] = 1.$$

Now $l.100 - l.101 = 2 - 2{,}0043214$

$$\left\{\begin{aligned} &= -1 + 3 - 2{,}0043214 \\ &= -1 + 0{,}9956786 \\ &= \bar{1}{,}9956786 \end{aligned}\right\}$$

hence, for the value of x,

$$x = -\frac{1}{0{,}0043214}$$
$$= -231 \text{ nearly.}$$

In fact,
$$\left(\frac{101}{100}\right)^x = \left(\frac{\frac{1}{101}}{100}\right)^{-x}$$
$$= \left(\frac{\frac{1}{101}}{\frac{1}{100}}\right)^{-x} = \left(\frac{100}{101}\right)^{-x}.$$

Now, from $\left(\frac{101}{100}\right)^x = 10$, we have $x = 231$; the exponent, therefore, to be given to the fraction $\frac{100}{101}$, when made equal to 10, is the same number 231, but taken with a negative sign.

For other examples the student may combine at pleasure several numbers, and make them equal to unknown quantities, and then resolve the equations by means of the logarithms. For instance, let a, b, c, d, e represent given numbers; we may form with them the following equations:

(1.) $\frac{a.b}{c.d} = ex.$

(2.) $a.b.c = d.e^x.$

(3.) $a^b.c.d = \frac{x}{e}.$

(4.) $\frac{1}{a.b.d} = \frac{c.e}{x}.$

(5.) $a^b.c^d = x^e;$

and so on; and applying the logarithms to them, we will find

(1.) $l.x = l.a + l.b - l.c - l.d - l.e.$

(2.) $x = \dfrac{l.a + l.b + l.c - l.d}{l.e}$.

(3.) $l.x = bl.a + l.c + l.d + l.e$.

(4.) $l.x = l.a + l.b + l.c + l.d + l.e$.

(5.) $l.x = \dfrac{bl.e + dl.c}{e}$.

With the exception of the second of these examples, in all the others we have not the value of x, but the value of the logarithm of x; now the corresponding number of any logarithm is given by the tables; hence, with the $l.x$, we may have x also.

It is plain, moreover, that any example like the preceding is the general formula of as many as the pupil will like, by substituting numbers for the symbols a, b, c, &c., and changing them at pleasure.

CHAPTER IV.

SERIES.

What algebraical series are, and their various orders.

§ 129. ARITHMETICAL progressions are the most simple of all algebraical series, and are called series of the first order.

The series of the second order are those whose second differences are equal, and the series of the third order are those whose third differences are equal, &c. That is, let

$$t',\ t'',\ t''',\ t^{iv},\ t^{v} \ldots$$

represent the numbers or terms of an algebraical series. If the differences

$$t'' - t',\ t''' - t'',\ t^{iv} - t''' \ldots.$$

between the successive numbers are all equal, the same number or terms belong to an algebraical series of the first order. But suppose the differences between the successive given terms to be unequal, so as to have

$$t''-t'=\tau',\ t'''-t''=\tau'',\ t^{\text{iv}}-t'''=\tau''' \ \ldots\ldots$$

If the second differences, or the differences between the terms τ', τ'', τ'''.... are equal, that is, if we have

$$\tau''-\tau'=\tau'''-\tau''=\tau^{\text{iv}}-\tau'''=\ldots.,$$

the given terms t', t'', t'''... then belong to an algebraic series of the second order. That is, the second differences also are unequal, and we have

$$\tau''-\tau'=\theta',\ \tau'''-\tau''=\theta'',\ \tau^{\text{iv}}-\tau'''=\theta''',\ \&c.;$$

but the third differences, or differences between the terms θ', θ'', θ'''.... are equal: the given terms t', t'', t'''.... belong in this case to a series of the third order. It is now easy to see when a series will be of the fourth, of the fifth, and generally of the m^{th} order.

It is likewise easy to infer from the foregoing remarks that the second, the third differences, and so on, of any algebraical series of the first order are all equal to zero, and the third, and following differences of any algebraical series of the second order are also equal to zero, and generally, the $(m-1)^{\text{th}}$ and following differences of any algebraical series of the m^{th} order are equal to zero.

Various questions concerning the series.

The most common investigations concerning these algebraical series are about the general term, and the sum of any number n of their first terms; but the principal object in view is that of reducing other functions to the form of a series.

This doctrine is copiously treated by modern writers, and with exquisite analysis in differential calculus.

It is not our intention to enter here into long discussions on the subject, and it will be enough for us to give an idea of it, treating briefly the first and second questions above mentioned.

General term of any series.

§ 180. Let us commence with the general term. The polynomial,

$$(p) = A + A_1 n + A_2 n^2 + \dots . + A_m n^m,$$

in which the coefficients A, A_1, $A_2 \dots A_m$ are constant quantities, represents the general term or the n^{th} term of any algebraic series of the m^{th} order. To have it demonstrated, it is enough to show that the m^{th} differences of the series corresponding to this term are all equal.

Now to say that (p) represents the n^{th} term of a series is the same as to say that it represents the first, the second term, and so on, when n is made equal to 1, to 2, &c., and the term immediately preceding the n^{th} will be obtained from (p), by changing in it n into $n - 1$. Now, call (p_1) the term preceding the n^{th}, it will be

$$(p_1) = A + A_1 (n - 1) + A_2 (n - 1)^2 + \dots + A_m (n - 1)^m,$$

and $(p) - (p_1)$ will be the first difference of any two successive terms of the series. But this difference—we may call it $(p_1)^1$—after reduction will take the following form:

$$B + B_1 n + B_2 n^2 + \dots + B_{m-1} n^{m-1};$$

hence,

$$(p_1)^1, \text{ or } (p) - (p_1) = B + B_1 n + B_2 n^2 + \dots + B_{m-1} n^{m-1};$$

in which difference, if we make n equal to 2, or 3, or 4, and so on, we will have the difference between the second and the third terms, between the third and the second, between the fourth and the third, and so on; we will have, also, the terms of another series, because $(p_1)^1$ has the same form as (p).

Repeating, therefore, on $(p_1)^1$ the same operation which we have made on (p)—that is, changing n into $n - 1$, to have the term immediately preceding $(p_1)^1$,—we may call it (p_2)—the difference will be

$$(p_2)^1, \text{ or } (p_1)^1 - (p_2) = C + C_1 n + C_2 n^2 + \dots . + C_{m-2} n^{m-2},$$

representing any one of the second differences of the series having (p) for general term, as it represents any one of the first differences between two successive terms of the series having for general term $(p_1)^1$.

It is now easy to see that the third differences of the series, corresponding to the general term (p), are given by

$$(p_3)^1, \text{ or } (p_2)^1 - (p_3) = D + D_1 n + D_2 n^2 + \dots . + D_{m-3} n^{m-3}, \text{ \&c.};$$

and the $(m - 1)^{th}$ differences by

$$(p_{m-1})^1, \text{ or } (p_{m-2})^1 - (p_{m-1}) = Q + Q_1 n;$$

and, finally, the m^{th} differences by

$$(p^m)^1, \text{ or } (p_{m-1})^1 - (p_m) = Q_1;$$

that is, any of the m^{th} differences of the series, whose general term is (p), is given by Q; that is, invariably by the same quantity; for whatever be n, Q_1 depends always equally on the constant coefficients A, A_1, A_2, ... of (p) only; hence it is constant, like them. The m^{th} differences, therefore, of the series corresponding to the term (p), are all equal; hence the same (p) is the general term of any series of the order m.

Nay, not only (p), but any expression reducible to the form of (p), represents likewise the general term of algebraic series of any order. Now, the following formula,

$$\left.\begin{array}{l} t_n = a_1 + a_2[n^2 - (n-1)^2] + a_3[n^3 - (n-1)^3] + \dots \\ \qquad \dots + a_{m+1}[n^{m+1} - (n-1)^{m+1}] \end{array}\right\} (o),$$

is reducible to the form of (p); hence (o) also represents the general term of any algebraic series.

Sum of any number of terms. § 131. From the same (o) we have the first, the second, the third term, and so on, of the series, by making in succession $n = 1$, $n = 2$, $n = 3$, &c., and these terms will be represented by the first member of (o), as follows:

$$t_1, t_2, t_3, \dots t_{n-1}, t_n.$$

Now, we say that the sum $t_1 + t_2 + \dots + t_n$ of these n terms, is given by

$$s_n = a_1 n + a_2 n^2 + a_3 n^3 + \dots + a_{m+1} n^{m+1} \} \; (o_1).$$

To demonstrate it, it is enough to show that (o_1) is equal to the first term t_1 of the series when n is made equal to 1, and equal to $t_1 + t_2$ when n is made equal to 2; equal to $t_1 + t_2 + t_3$ when n is made equal to 3, &c. Now that (o_1) is equal to t_1 when n is made equal to 1, is evident by observing that (o) is equal to (o_1), when in both of them we make $n = 1$. Before showing that (o_1) is equal to $t_1 + t_2$, when n is made equal to 2, &c., observe that if in (o_1) we change n into $n-1$, we will have

$$s_{n-1} = a_1(n-1) + a_2(n-)^2 + a_3(n-1)^3 + \dots + a_{m+1}(n-1)^{m+1};$$

hence,

$$\begin{array}{l} s_n - s_{n-1} = a_1 + a_2[n^2 (n-1)^2] + a_3[n^3 - (n-1)^3] + \dots \\ \qquad + a_{m+1}[n^{m+1} - (n-1)^{m+1}]; \end{array}$$

but this last member is the general term (o) of the series; hence

$$s_n - s_{n-1} = t_n \ (o_2);$$

and consequently if in (o_1) we make $n = 2$, we will have

$$s_2 - s_1 = t_2;$$

but $s_1 = t_1$; hence $\quad s_2 = t_1 + t_2.$

If in (o_1) we make $n = 3$, then from the same (o_2) we have

$$s_3 - s_2 = t_3;$$

but $s_2 = t_1 + t_2$; hence

$$s_3 = t_1 + t_2 + t_3.$$

And if in (o_1) we make $n = 4$, we will evidently have

$$s_4 = t_1 + t_2 + t_3 + t_4;$$

and generally, s_n or

$$a_1 n + a_2 n^2 + a_3 n^3 + \dots = t_1 + t_2 + \dots + t_n;$$

that is, (o_1) is the expression of the sum of n terms of any series of the m^{th} order.

Let us now pass to see how, by means of the formulas (o) and (o_1), we may find the general term and the sum of some given series.

Examples. First. § 182. Let 3, 6, 10, 15, 21 be the first terms of a given series, in which the second differences are all equal to 1; hence, in the general term of this series, m must be equal to 2; that is, the formula (o) will be, in this case,

$$t_n = a_1 + a_2[n^2 - (n-1)^2] + a_3[n^3 - (n-1)^3].$$

To find out the coefficients a_1, a_2, a_3, make in succession $n = 1$, $n = 2$, $n = 3$; and since, with these substitutions, the general term ought to represent the first three terms of the given series, we will have the equations,

$$a_1 + a_2 + a_3 = 3,$$
$$a_1 + 3a_2 + 7a_3 = 6,$$
$$a_1 + 5a_2 + 19a_3 = 10,$$

from which $\quad a_3 = \frac{1}{6},\ a_2 = 1,\ a_1 = \frac{11}{6}.$

Hence, the general term of the given series is

$$t_n = \frac{11}{6} + 2n - 1 + \frac{1}{6}(3n^2 - 3n + 1),$$
$$= 1 + \frac{3}{2}n + \frac{1}{2}n^2,$$
$$= \frac{(n+1)(n+2)}{2};$$

and the sum of n terms,

$$s_n = \frac{11}{6}n + n^2 + \frac{1}{6}n^3.$$

Second. Let also, 1, 5, 14, 30, 55, 91 . . . be the first terms of another series whose third differences are all equal to 2. Making, therefore, in (o) $m = 3$, we will have for the general term of this given series

$$t_n = a_1 + a_2[n^2-(n-1)^2] + a^3[n^3-(n-1)^3] + a_4[n^4-(n-1)^4],$$

from which, making in succession $n = 1$, $n = 2$, $n = 3$, $n = 4$, we will have the equations,

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &= 1,\\ a_1 + 3a_2 + 7a_3 + 15a_4 &= 5,\\ a_1 + 5a_2 + 19a_3 + 65a_4 &= 14,\\ a_1 + 7a_2 + 37a_3 + 175a_4 &= 30; \end{aligned}$$

and from these the following values of the coefficients:

$$a_1 = \frac{1}{6},\ a_2 = \frac{5}{12},\ a_3 = \frac{1}{3},\ a_4 = \frac{1}{12};$$

and substituting these values in the general term, we have

$$t_n = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3;$$

and for the sum of the first n terms of the same given series, we will find

$$s_n = \frac{1}{6}n + \frac{5}{12}n^2 + \frac{1}{3}n^3 + \frac{1}{12}n^4.$$

Other examples. Some of the coefficients a_2, a_3 . . of (o), and the first term a_1 also, may be equal to zero, or may be such that some terms of (o) evolved be mutually eliminated. In this supposition the general term may apparently have a different form from that of (p). So, for example, we may have

$$t_n = n^2,\ t_n = n^3.$$

Suppose now that such general terms are given, we may obtain the sums also; for substituting successively the natural numbers 1, 2, 3 . . . instead of n, from $t_n = n^2$, we have the series

1, 4, 9, 16, 25 . . .,

and from $t_n = n^3$, 1, 8, 27, 64, 125 . . . .,

the first of which has the second differences, and the last the third differences constant; hence, with the same process followed in the preceding examples, we will find for the first

$$a_1 = \frac{1}{6},\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{3};$$

and for the second,

$$a_1 = 0,\ a_2 = \frac{1}{4},\ a_3 = \frac{1}{3},\ a_4 = \frac{1}{4};$$

and, therefore, the sum of n terms of the series having $t_n = n^2$ for its general term, is

$$s_n = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3;$$

and the sum of n terms of the series having $t_n = n^3$ for its general term, is

$$s_n = \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4.$$

PROBLEMS.

1st. Two merchants, A and B, possess a capital of 38700 pounds, but the capital of A is twice that of B; how much does each one of them possess? Ans. A.... 25800, B....12900.

2d. Philip makes a present to his children, M and N, of 2500 dollars, but M gets as many times 20 dollars as N gets 5. What is the share of each? Ans. M.... 2000, N.... 500.

3d. Divide the number 237 in two such parts, that the first be greater than the second by one quarter of the second.

Ans. 1st. $131 + \frac{2}{3}$, 2d. $105 + \frac{1}{3}$.

4th. Two friends wish to buy a horse, but the first cannot pay but one-fifth of the price, and the second one-seventh only; to have the horse they should add £20. What is the price of the horse?

Ans. $x = 30 + \frac{10}{23}$

5th. A merchant after his speculations finds that he has gained 15 per cent. on his capital, and the amount of his actual fortune is £15571. What was the original capital? Ans. $x = 13540$.

6th. A man sells a certain amount of goods in three successive days. The first day he loses one-sixth of the value of the articles he is going to sell during the three days; the second day he loses one-tenth of the same value; but the last day he gains one-third of the price. At the end he finds that he has gained no more than three dollars. What is the price of the articles sold in the three days?

Ans. $x = 45$.

7th. Twice the number of years of my age, diminished by the fourth of the same number, gives twelve years more than those of my age What is my age? Ans. $x = 16$.

8th. A father sends to his five children 1000 dollars, with the condition that the eldest should have 20 dollars more than the second, and the second 20 dollars more than the third, and so likewise the rest. How many shall the first of the children have?

Ans. $x = 240$.

9th. I had 42 shillings, and I paid a part of them. If you divide the remainder by the number of those which I have paid, you will have 12. How many did I pay. Ans. $x = 3 + \frac{3}{13}$.

10th. Two travellers go from the same place to another. But the first, who travels 12 miles per day, leaves the place ten days before the second. The second travels 27 miles per day. After how many days shall the second reach the first? Ans. $x = 8$.

11th. A mortar throws on a fortress 86 shells, before a second mortar begins to throw its own. The second mortar throws 7 shells in the same time in which the first throws 8. But the quantity of gunpowder consumed in three explosions by the second, is the same as that consumed by the first in four explosions. How many bombs must the second mortar throw on the fortress to consume the same quantity of gunpowder as the first? Ans. $x = 189$.

12th. A friend of mine 40 years old, has a son 10 years old. How many years shall pass before the age of the father be double that of the son? Ans $x = 20$.

13th. Give me the expression of two numbers whose sum is a, and the sum of the product of the first by m, and of the second by n is b.

Ans. $\frac{b - an}{m - n}, \frac{ma - b}{m - n}$.

14th. A general wishes to range his regiment in a square battalion; he tries two ways, in the first of which there remain 80 men, besides the full square; in the second, which consists in adding a man to each rank; he finds that there are 50 men wanting to finish the square. How many men does the regiment contain? Ans. $x = 1975$.

15th. Find such a number, that adding to it in succession a and b, and squaring the sums, the difference of these sums be d.

Ans. $x = \frac{d + b^2 - a^2}{2(a - b)}$.

16th. Find two numbers, whose sum is 87, and their difference 13.

Ans. $x = 50$, $y = 37$.

17th. The first of three friends A, B, C gives to B and C so many of his own dollars as to redouble their original number. B then redoubles in like manner the money of A and C, and finally C redoubles in his turn the money of A and B. After this, they find that each one of them has 16 dollars. What was the original number x of A, and the original numbers y and z of B and C?

Ans. $x = 26$, $y = 14$, $z = 8$.

18th. I have two boxes with money in them. If I add 8 pieces to those in the first box, the pieces contained in the first will be exactly one-half of those contained in the second. But if, instead of adding the 8 pieces to those of the first box, I put them in the second, the pieces of the second will be three times those of the first. How many pieces does each box contain? Ans. $x = 24$, $y = 64$.

19th. The money of A and that of B make £570. If the first would have three times, and the second five times more money, the money of both would amount to £2350. How many are the pounds of A? how many those of B? Ans. $x = 250$, $y = 320$.

20th. Two baskets contain some dozen of apples. If those of the first basket are sold at 5 cents a dozen, and those of the second at ten cents, all will be sold for two dollars. But if the apples of the first basket be sold at ten cents, and those of the second at five cents a dozen, they will be sold for two dollars and 50 cents. How many dozen of apples are in the first? how many in the second basket?

Ans. $x = 20$, $y = 10$.

21st. Some students go on an excursion. If they were five more and each would pay 1 dollar more, the expense would be 61½ dollars more; but if they were three less, and each would pay 1½ dollars less, the expense would be 42 dollars less. How many are the students, and what is their fare? Ans. St. $x = 14$, F. $y = 8\frac{1}{2}$.

22d. Find two numbers whose sum is m times, and whose product n times as great as their difference.

Ans. $x = \frac{2n}{m-1}$, $y = \frac{2n}{m+1}$.

23d. The sum of two numbers is a, and the difference of their squares is b. What are these numbers? Ans. $x = \frac{a^2+b}{2a}$, $y = \frac{a^2-b}{2a}$.

24th. Add x to 94, and then subtract the same x from 94. The product of the sum by the difference, gives 8512. What is the value of x? Ans. $x = 18$.

25th. If the third part of a number be multiplied by the fourth part of the same number, and this product be added to that of the same number multiplied by 5, we will have a result so much above 200 as the same number is below 280. What is the number?

Ans. $x = 48$.

26th. One of two brothers is 20 years older than the other, and if the age of the first be multiplied by that of the second, the product will be 2500 years more than the sum of the years of each of the two brothers. How old is the younger?

Ans. $x = 42$.

27th. Two boys sell 100 melons. The first sells his part at a price different from that at which the second sells his. And yet they obtain the same price. But if the first should have the melons of the second, and *vice versâ*, the first selling them at his own price would gain 15 dollars, and the second $6\frac{2}{3}$ dollars. How many melons has the first boy?

Ans. $x = 40$.

28th. Find two numbers whose product is 750, and whose quotient is $3\frac{1}{3}$.

Ans. $x = 15$, $y = 50$.

29th. Find the expression of two numbers whose product is a, and whose quotient is b.

Ans. $x = \sqrt{\frac{a}{b}}$, $y = \sqrt{ab}$.

30th. Find two such numbers that the sum of their squares be 13001, and the difference of the same squares be 1449.

Ans. $x = 85$, $y = 76$.

THE END.